

Axiomatization of the Choquet integral for 2-dimensional heterogeneous product sets

Mikhail Timonin^a

^a*Queen Mary University of London
Mile End Road London, UK, E1 4NS
+442078826139*

Abstract

We prove a representation theorem for the Choquet integral model. The preference relation is defined on a two-dimensional heterogeneous product set $X = X_1 \times X_2$ where elements of X_1 and X_2 are not necessarily comparable with each other. However, making such comparisons in a meaningful way is necessary for the construction of the Choquet integral (and any rank-dependent model). We construct the representation, study its uniqueness properties, and look at applications in multicriteria decision analysis, state-dependent utility theory, and social choice. Previous axiomatizations of this model, developed for decision making under uncertainty, relied heavily on the notion of comonotonicity and that of a “constant act”. However, that requires X to have a special structure, namely, all factors of this set must be identical. Our characterization does not assume commensurateness of criteria a priori, so defining comonotonicity becomes impossible.

Keywords: Choquet integral, Decision theory, MCDA

1. Introduction

Rank-dependent models appeared in axiomatic decision theory in reply to the criticism of Savage’s postulates of rationality (Savage, 1954). The renowned Ellsberg paradox (Ellsberg, 1961) has shown that people can violate Savage’s axioms and still consider their behaviour rational. First models accounting for the so-called uncertainty aversion observed in this paradox appeared in the 1980s, in the works of Quiggin (1982) and others (see (Wakker, 1991a) for a review). One particular generalization of the expected utility model (EU) characterized by Schmeidler (1989) is the Choquet expected utility (CEU), where probability is replaced by a non-additive set function (called capacity) and integration is performed using the Choquet integral.

Since Schmeidler’s paper, various versions of the same model have been characterized in the literature (e.g. (Gilboa, 1987; Wakker, 1991b)). CEU has gained some momentum in both theoretical and applied economic literature, being used mainly for analysis of problems involving Knightian uncertainty. At the same time, rank-dependent models, in particular the Choquet integral, were adopted in multiattribute utility theory (MAUT) (Keeney and Raiffa, 1976). Here the integral gained popularity due to the tractability of non-additive measures in this context (see (Grabisch and Labreuche, 2008) for a review).

URL: m.timonin@qmul.ac.uk (Mikhail Timonin)

The model permitted various preferential phenomena, such as criteria interaction, which were impossible to reflect in the traditional additive models.

The connection between MAUT and decision making under uncertainty has been known for a long time. In the case when the number of states is finite, which is assumed hereafter, states can be associated with criteria. Accordingly, acts correspond to multicriteria alternatives. Finally, the sets of outcomes at each state can be associated with the sets of criteria values. However, this last transition is not quite trivial. It is commonly assumed that the set of outcomes is the same in each state of the world (Savage, 1954; Schmeidler, 1989). In multicriteria decision making the opposite is true. Indeed, consider preferences of consumers choosing cars. Each car is characterized by a number of features (criteria), such as colour, maximal speed, fuel consumption, comfort, etc. Apparently, sets of values taken by each criterion can be completely different from those of the others. In such context the ranking stage of rank-dependent models, which in decision under uncertainty involves comparing outcomes attained at various states, would amount to comparing colours to the level of fuel consumption, and maximal speed to comfort.

Indeed, the traditional additive model (Debreu, 1959; Krantz et al., 1971) only implies meaningful comparability of *units* between goods in the bundle, but not of their absolute levels. However, in rank-dependent models such comparability seems to be a necessary condition. This paper develops a characterization of the Choquet integral for two-dimensional sets with comparability (commensurateness) of the criteria not assumed a priori.

Let $X = X_1 \times X_2$ be a (heterogeneous) product set and \succsim a binary relation defined on this set. In multiattribute utility theory, elements of the set X are interpreted as alternatives characterized by two criteria taking values from sets X_1 and X_2 . In decision making under uncertainty, the factors of the set X usually correspond to outcomes in various states of the world, and an additional restriction $X_1 = X_2 = Y$ is being made. Thus in CEU, the set X is homogeneous, i.e. $X = Y^n$.

Previous axiomatizations of the Choquet integral have been given for this special case of $X = Y^n$ (see (Köbberling and Wakker, 2003) for a review of approaches) and its variant $X = \mathbb{R}^n$ (see (Grabisch and Labreuche, 2008) for a review). Another approach using conditions on the utility functions was proposed in (Labreuche, 2012). A conjoint axiomatization of the Choquet integral for the case of a general X was an open problem in the literature. One related result that should be mentioned is the recent conjoint axiomatization of another non-additive integral, the Sugeno integral ((Greco et al., 2004; Bouyssou et al., 2009)).

The crucial difference between our result and previous axiomatizations is that the notions of “comonotonicity” and “constant act” are no longer available in the heterogeneous case. Recall that two acts are called comonotonic in CEU if their outcomes have the same ordering. A constant act is plainly an act having the same outcome in every state of the world. Apparently, since criteria sets X_1 and X_2 in our model can be completely disjoint, neither of the notions can be used anymore due to the fact that there does not exist a meaningful built-in order between elements of sets X_1 and X_2 . New axioms and proof techniques must be introduced to deal with this complication.

The paper is organized as follows. Section 2 defines the Choquet integral and looks at its properties. Section 3 states and explains the axioms. Section 4 gives the representation theorem. Section 5 discusses the main result and its economic interpretations. The proof

of the theorem is presented in the Appendix; in particular, necessity of axioms is discussed in section A.11.

2. Choquet integral in MAUT

Let $N = \{1, 2\}$ be a set (of criteria) and 2^N its power set.

Definition 1. Capacity (non-additive measure, fuzzy measure) is a set function $\nu : 2^N \rightarrow \mathbb{R}_+$ such that:

1. $\nu(\emptyset) = 0$;
2. $A \subseteq B \Rightarrow \nu(A) \leq \nu(B)$, $\forall A, B \in 2^N$.

In this paper, it is also assumed that capacities are normalized, i.e. $\nu(N) = 1$.

Definition 2. The Choquet integral of a function $f : N \rightarrow \mathbb{R}$ with respect to a capacity ν is defined as

$$C(\nu, f) = \int_0^\infty \nu(\{i \in N : f(i) \geq r\}) dr + \int_{-\infty}^0 [\nu(\{i \in N : f(i) \geq r\}) - 1] dr$$

Denoting the range of $f : N \rightarrow \mathbb{R}$ as (f_1, \dots, f_n) , the definition can be expressed as:

$$C(\nu, (f_1, \dots, f_n)) = \sum_{i=1}^n (f_{(i)} - f_{(i-1)}) \nu(\{j \in N : f_j \geq f_{(i)}\})$$

where $f_{(1)}, \dots, f_{(n)}$ is a permutation of f_1, \dots, f_n such that $f_{(1)} \leq f_{(2)} \leq \dots \leq f_{(n)}$, and $f_{(0)} = 0$.

2.1. The model

Let \succsim be a binary relation on the set $X = X_1 \times X_2$. $\succ, \prec, \preceq, \sim, \not\sim$ are defined in the usual way. We say that \succsim can be represented by a Choquet integral, if there exists a capacity ν and functions $f_1 : X_1 \rightarrow \mathbb{R}$ and $f_2 : X_2 \rightarrow \mathbb{R}$, called value functions, such that:

$$x \succsim y \iff C(\nu, (f_1(x_1), f_2(x_2))) \geq C(\nu, (f_1(y_1), f_2(y_2))).$$

As seen in the definition of the Choquet integral, its calculation involves comparison of the f_i 's to each other. It is not immediately obvious how this operation can have any meaning in the MAUT context. It is well-known that comparing *levels* of value functions for various attributes is meaningless in the additive model (Krantz et al., 1971) (recall that the origin of each value function can be changed independently). In the homogeneous case $X = Y^n$ this problem is readily solved, as we have a single set of outcomes Y (in the context of decision making under uncertainty). The required order is either assumed as given (Wakker, 1991a) or is readily derived from the ordering of the constant acts (α, \dots, α) (Wakker, 1991b). Since there is a single outcome set, we also have a single value (utility) function $U : Y \rightarrow \mathbb{R}$, and thus comparing $U(y_1)$ to $U(y_2)$ is perfectly sensible, since U represents the order on the set Y . None of these methods can be readily applied in the heterogeneous case.

2.2. Properties of the Choquet integral

Below are given some important properties of the Choquet integral:

1. Functions $f : N \rightarrow \mathbb{R}$ and $g : N \rightarrow \mathbb{R}$ are comonotonic if for no $i, j \in N$ we have $f(i) > f(j)$ and $g(i) < g(j)$. For all comonotonic f the Choquet integral reduces to the Lebesgue integral. In the finite case, the integral is accordingly reduced to a weighted sum.
2. Particular cases of the Choquet integral (e.g. (Grabisch and Labreuche, 2008)).
 - If $\nu(\{1\}) = \nu(\{2\}) = 1$, then $C(\nu, (f_1, f_2)) = \max(f_1, f_2)$.
 - If $\nu(\{1\}) = \nu(\{2\}) = 0$, then $C(\nu, (f_1, f_2)) = \min(f_1, f_2)$.
 - If $\nu(\{1\}) + \nu(\{2\}) = 1$, then $C(\nu, (f_1, f_2)) = \nu(\{1\})f_1 + \nu(\{2\})f_2$

Property 1 states that the set X can be separated into subsets corresponding to particular orderings of the value functions. In the case of two criteria there are only two such sets: $\{x \in X : f_1(x_1) \geq f_2(x_2)\}$ and $\{x \in X : f_2(x_2) \geq f_1(x_1)\}$. Since the integral on each of the sets is reduced to a weighted sum, i.e. an additive representation, we should expect many of the axioms of the additive conjoint model to be valid on this subsets. This is the intuition behind several of the axioms given in the following section.

3. Axioms

Definition 3. A relation \succsim on $X_1 \times X_2$ satisfies triple cancellation, provided that for every $a, b, c, d \in X_1$ and $p, q, r, s \in X_2$, if $ap \preceq bq$, $ar \succsim bs$, and $cp \succsim dq$, then $cr \succsim ds$.

Definition 4. A relation \succsim on $X_1 \times X_2$ is independent, iff for $a, b \in X_1$, $ap \succsim bp$ for some $p \in X_2$ implies that $aq \succsim bq$ for every $q \in X_2$; and, for $p, q \in X_2$, $ap \succsim aq$ for some $a \in X_1$ implies that $bp \succsim bq$ for every $b \in X_1$.

A1. \succsim is a weak order.

A2. Weak separability - for any $a_i p_j, b_i p_j \in X$ such that $a_i p_j \succ b_i p_j$, we have $a_i q_j \succ b_i q_j$ for all $q \in X_j$, for $i, j \in \{1, 2\}$.

The separability condition is weaker than the one normally used.¹ In fact, it only rules out a reversal of strict preference. Note, that the condition implies that for any $a, b \in X_1$ either $ap \succsim bp$ or $bp \succ ap$ for all $p \in X_2$ (symmetrically for the second coordinate). Apparently, transitivity also holds: if $ap \succsim bp$ for all $p \in X_2$ and $bp \succsim cp$ for all $p \in X_2$, then $ap \succsim cp$ for all $p \in X_2$. This allows to introduce the following weak orders:

Definition 5. For all $a, b \in X_1$ define \succsim_1 as $a \succsim_1 b \iff ap \succsim bp$ for all $p \in X_2$. Define \succsim_2 symmetrically.

Definition 6. We call $a \in X_1$ minimal if $b \succsim_1 a$ for all $b \in X_1$, and maximal if $a \succsim_1 b$ for all $b \in X_1$. Symmetric definitions hold for X_2 .

¹The condition first appeared in (Bliss, 1975), and in this form in (Mak, 1984)

Definition 7. For any $z \in X$ define $\mathbf{SE}_z = \{x: x \in X, x_1 \succ_1 z_1 \text{ and } z_2 \succ_2 x_2\}$, and $\mathbf{NW}_z = \{x: x \in X, x_2 \succ_2 z_2 \text{ and } z_1 \succ_1 x_1\}$.

The “rectangular” cones \mathbf{SE}_z and \mathbf{NW}_z play a significant role in the sequel.

A3. For any $z \in X$, triple cancellation holds either on \mathbf{SE}_z or on \mathbf{NW}_z .

The axiom says that the set X can be covered by “rectangular” cones, such that triple cancellation holds within each cone. We will call such cones “3C-cones”. The axiom effectively divides X into subsets, defined as follows.

Definition 8. We say that

- $x \in \mathbf{SE}$ if:
 - There exists $z \in X$ such that z_1 is not maximal and z_2 is not minimal, triple cancellation holds on \mathbf{SE}_z , and $x \in \mathbf{SE}_z$, or
 - x_1 is maximal or x_2 is minimal and for no $y \in \mathbf{SE}_x \setminus x$ triple cancellation holds on \mathbf{NW}_x ;
- $x \in \mathbf{NW}$ if:
 - There exists $z \in X$ such that z_1 is not maximal and z_2 is not minimal, triple cancellation holds on \mathbf{SE}_z , and $x \in \mathbf{SE}_z$, or
 - x_1 is minimal or x_2 is maximal and for no $y \in \mathbf{NW}_x \setminus x$ triple cancellation holds on \mathbf{SE}_x .

Define also $\Theta = \mathbf{NW} \cap \mathbf{SE}$.

Definition 9. We say that $i \in N$ is essential on $A \subset X$ if there exist $x_i x_j, y_i x_j \in A$, $i, j \in N$, such that $x_i x_j \succ y_i x_j$.

Essentiality of coordinates is discussed in detail in Section A.3.

A4. Whenever $ap \preceq bq, ar \succ bs, cp \succ dq$, we have that $cr \succ ds$, provided that either:

- a) $ap, bq, ar, bs, cp, dq, cr, ds \in \mathbf{NW}(\mathbf{SE})$, or;
- b) $ap, bq, ar, bs \in \mathbf{NW}$ and $i = 2$ is essential on \mathbf{NW} and $cp, dq, cr, ds \in \mathbf{SE}$ or vice versa, or;
- c) $ap, bq, cp, dq \in \mathbf{NW}$ and $i = 1$ is essential on \mathbf{NW} and $cr, ds \in \mathbf{SE}$ or vice versa.

Informally, the meaning of the axiom is that ordering between preference differences (“intervals”) is preserved irrespective of the “measuring rods” used to measure them. However, contrary to the additive case this does not hold on all X , but only when either points involved in all four relations lie in a single 3C-cone, or points involved in two relations lie in one 3C-cone and those involved in the other two in another.

A5. Whenever $ap \preceq bq, cp \succcurlyeq dq$ and $ay_0 \sim x_0\pi(a), by_0 \sim x_0\pi(b), cy_1 \sim x_1\pi(c), dy_1 \sim x_1\pi(d)$, and also $e\pi(a) \succcurlyeq f\pi(b)$, we have $e\pi(c) \succcurlyeq f\pi(d)$, for all $ap, bq, cp, dq \in \mathbf{NW}$ or \mathbf{SE} provided coordinate $i = 1$ is essential on the subset which contains these points, $ay_0, by_0, cy_1, dy_1 \in \mathbf{NW}$ or \mathbf{SE} , $x_0\pi(a), x_0\pi(b), x_1\pi(c), x_1\pi(d) \in \mathbf{NW}$ or \mathbf{SE} provided coordinate $i = 2$ is essential on the subset which contains these points, $e\pi(a), f\pi(b), e\pi(c), f\pi(d) \in \mathbf{NW}$ or \mathbf{SE} . Same condition holds for the other dimension symmetrically.

The formal statement of **A5** is rather complicated, but it simply means that the ordering of the intervals is preserved across dimensions. Together with **A4** the conditions are similar to Wakker's trade-off consistency condition (Wakker, 1991a). The axiom bears even stronger similarity to Axiom 5 (compatibility) from section 8.2.6 of (Krantz et al., 1971). Roughly speaking, it says that if the interval between c and d is larger than that between a and b , then projecting these intervals onto another dimension by means of the equivalence relations must leave this order unchanged. We additionally require the comparison of intervals and projection to be consistent - meaning that quadruples of points in each part of the statement lie in the same 3C-cone. Another version of this axiom, which is going to be used frequently in the proofs, is formulated in terms of standard sequences in Lemma 3.

A6. Bi-independence : Let $ap, bp, cp, dp \in \mathbf{SE}(\mathbf{NW})$ and $ap \succ bp$. If for some $q \in X_2$ also exist $cq \succ dq$, then $cp \succ dp$. Symmetric condition holds for the second coordinate.

This axiom is similar to "strong monotonicity" in (Wakker, 1991a). We analyze its necessity and the intuition behind it in section A.3.

A7. Both coordinates are essential on X .

A8. Restricted solvability : if $x_ia_j \succcurlyeq y \succcurlyeq x_ic_j$, then there exists $b : x_ib_j \sim y$, for $i, j \in \{1, 2\}$.

A9. Archimedean axiom: for every $z \in \mathbf{NW}(\mathbf{SE})$ every bounded standard sequence contained in $\mathbf{NW}_z(\mathbf{SE}_z)$ is finite.

Structural assumption.. For no $a, b \in X_1$ we have $ap \sim bp$ for all $p \in X_2$. Similarly, for no $p, q \in X_2$ we have $ap \sim aq$ for all $a \in X_1$. If such points exist, say $ap \sim bp$ for all $p \in X_2$, then we can build the representation for a set $X'_1 \times X_2$ where $X'_1 = X_1 \setminus a$, and later extend it to X by setting $f_1(a) = f_1(b)$.

X is order dense.. Whenever $x \succ y$ there exists z such that $x \succ z \succ y$. From this and restricted solvability immediately follows that \succcurlyeq_i is order dense as well, in other words, whenever $a_ip_j \succcurlyeq b_ip_j$ there exists $c \in X_i$ such that $a_ip_j \succ c_ip_j \succ b_ip_j$, for $i, j \in N$.

"Closedness" of SE and NW.. Finally, we *extend* the set X as follows. Whenever exist $ap \notin \mathbf{NW}$ and $bp \notin \mathbf{SE}$, there exist also $cp \in \Theta$. Similarly, whenever exist $ap \notin \mathbf{NW}$ and $aq \notin \mathbf{SE}$, there exist also $ar \in \Theta$.

3.1. Discussion of axioms

Roughly speaking, for two dimensional sets the Choquet integral can be characterized by saying that X is divided into two subsets such that \succsim on each of them has an additive representation, while the intersection of these subsets (in the representation) is the line $\{x : f_1(x_1) = f_2(x_2)\}$. In the previous characterizations locating these subsets was straightforward, as they are nothing else but the comonotonic subsets of X . In this paper we take a different approach. Instead, we state that X can be separated into two subsets without imposing any additional constraints on their location and then use additional axioms to characterize the intersection of these subsets and to show that it is mapped to the line $\{x : f_1(x_1) = f_2(x_2)\}$.

Our axioms aim to reflect the main properties of the Choquet integral. The first one is that the set X can be divided into subsets, such that within every such subset the preference relation can be represented by an additive function. The axiom **(A3)** we introduce is similar to the “2-graded” condition previously used for characterizing of MIN/MAX and the Sugeno integral ((Greco et al., 2004; Bouyssou et al., 2009)). At every point $z \in X$ it is possible to build two “rectangular cones”: $\{x : x_1 \succsim_1 z_1 \text{ and } z_2 \succsim_2 z_2\}$, and $\{x : x_2 \succsim_2 z_2 \text{ and } z_1 \succsim_1 z_1\}$. The axiom states that triple cancellation must then hold on at least one of these cones.

The second property is that the additive representations on different subsets are interrelated, in particular trade-offs between criteria values are consistent across subsets both within the same dimension and for different ones. This is reflected by two axioms **(A4, A5)**, similar to the ones used in (Wakker, 1991b) and (Krantz et al., 1971) (section 8.2). One, roughly speaking, states that triple cancellation holds across cones, while the other says that the ordering of intervals on any dimension must be preserved when they are projected onto another dimension by means of equivalence relations.

These axioms are complemented by a new condition called bi-independence **(A6)** and weak separability **(A2)** - which together reflect the monotonicity property of the integral.

Standard essentiality, “comonotonic” Archimedean axiom and restricted solvability **(A7, A8, A9)** complete the list. Finally, \succsim is supposed to be a weak order, and X is order dense **(A1)**.

Our most important axioms - **A3, A4, A5, A6**, are not only sufficient, but also necessary. Necessity and detailed analysis of **A6** is given in Section A.3, necessity of **A4** and **A5** is proved in Section A.11, whereas necessity of **A3** is immediate (in the representation one of the regions \mathbf{NW}_z and \mathbf{SE}_z is necessarily contained in a comonotonic subset of \mathbb{R}^2). Necessity of some of the remaining axioms is well-known Wakker (1991a); Bouyssou and Pirlot (2004).

4. Representation theorem

Theorem 1. *Let \succsim be an order on X and let X be order dense and the structural assumption hold. Then, if axioms **A1-A9** are satisfied, there exists a capacity ν and value functions $f_1 : X_1 \rightarrow \mathbb{R}$, $f_2 : X_2 \rightarrow \mathbb{R}$, such that \succsim can be represented by the Choquet integral:*

$$x \succsim y \iff C(\nu, (f_1(x_1), f_2(x_2))) \geq C(\nu, (f_1(y_1), f_2(y_2))), \quad (1)$$

for all $x, y \in X$. Moreover, ν is determined uniquely and value functions have the following uniqueness properties:

1. If $\nu(\{1\}) + \nu(\{2\}) = 1$, then for any functions $g_1 : X_1 \rightarrow \mathbb{R}, g_2 : X_2 \rightarrow \mathbb{R}$ such that (1) holds with f_i substituted by g_i , we have $f_i(x_i) = \alpha g_i(x_i) + \beta_i$ for some $\alpha > 0$.
2. If $\nu(\{1\}) \in (0, 1)$ and $\nu(\{2\}) \in (0, 1)$ and $\nu(\{1\}) + \nu(\{2\}) \neq 1$, then for any functions $g_1 : X_1 \rightarrow \mathbb{R}, g_2 : X_2 \rightarrow \mathbb{R}$ such that (1) holds with f_i substituted by g_i , we have $f_i(x_i) = \alpha g_i(x_i) + \beta$ for some $\alpha > 0$.
3. If $\nu(\{2\}) \in (0, 1), \nu(\{1\}) \in \{0, 1\}$, then for any functions $g_1 : X_1 \rightarrow \mathbb{R}, g_2 : X_2 \rightarrow \mathbb{R}$ such that (1) holds with f_i substituted by g_i , we have :
 - $f_i(x_i) = \alpha g_i(x_i) + \beta$, for all x such that $f_1(x_1) < \max f_2(x_2)$ and $f_2(x_2) > \min f_1(x_1)$;
 - $f_i(x_i) = \psi_i(g_i(x_i))$ where ψ_i is an increasing function, otherwise.
4. If $\nu(\{2\}) \in \{0, 1\}, \nu(\{1\}) \in (0, 1)$, then for any functions $g_1 : X_1 \rightarrow \mathbb{R}, g_2 : X_2 \rightarrow \mathbb{R}$ such that (1) holds with f_i substituted by g_i , we have :
 - $f_i(x_i) = \alpha g_i(x_i) + \beta$, for all x such that $f_2(x_2) < \max f_1(x_1)$ and $f_1(x_1) > \min f_2(x_2)$;
 - $f_i(x_i) = \psi_i(g_i(x_i))$ where ψ_i is an increasing function, otherwise.
5. If $\nu(\{1\}) = \nu(\{2\}) = 0$ or $\nu(\{1\}) = \nu(\{2\}) = 1$, then for any functions $g_1 : X_1 \rightarrow \mathbb{R}, g_2 : X_2 \rightarrow \mathbb{R}$ such that (1) holds with f_i substituted by g_i , we have : $f_i(x_i) = \psi_i(g_i(x_i))$ where ψ_i are increasing functions such that $f_1(x_1) = f_2(x_2) \iff g_1(x_1) = g_2(x_2)$.

4.1. Uniqueness properties imply commensurateness

As the uniqueness part of Theorem 1 states, unless \succsim can be represented by an additive functional on all of X (Case 1), the representation implies commensurateness of *levels* of utility functions defined on different factors of the product set. Indeed, we have that if $f_1(x_1) \geq f_2(x_2)$ in one representation, then necessarily $g_1(x_1) \geq g_2(x_2)$ in another one. This is a much stronger uniqueness result in comparison to the traditional additive models. In Section 5 we discuss some economic implications of this.

4.2. Extension to n dimensions

This paper provides a characterization of the Choquet integral for two-dimensional sets, which allows to have simpler proofs. We believe that an extension to n dimensions is mostly a technical task. Axiom **A3** would be separated into two conditions. One is similar to the current **A3**, and holds for any pair of dimensions with the remaining coordinates fixed, and the other is acyclicity of the absence of additivity on the n -criteria counterparts of regions \mathbf{NW}_z and \mathbf{SE}_z in between pairs of coordinates. Just as in the present paper, stronger uniqueness would be due to the lack of additivity. The remaining differences are technical.

5. Applications

5.1. Multicriteria decision analysis

In multicriteria context our result implies that the decision maker constructs a one-to-one mapping between elements of criteria sets (their subsets to be precise). Some authors interpret this by saying that criteria elements sharing the same utility values present the same level of “satisfaction” for the decision maker (Grabisch and Labreuche, 2008). Technically, such statements are meaningful, in the sense that permissible scale transformations do not render them ambiguous or incorrect, unless the representation is additive. However, the substance of statements like “ x_1 on criterion 1 is at least as good as x_2 on criterion 2” (which would correspond to $f_1(x_1) \geq f_2(x_2)$) is not easy to grasp. Perhaps, it is possible to think about workers performing various tasks within a single project. From the perspective of a project manager, achievements of various workers, serving as criteria in this example, can be level-comparable despite being physically different, if the project has global milestones (i.e. scale) which are mapped to certain personal milestones for every involved person. The novelty of our characterization is that this scale is not given a priori. Instead, we only observe preferences of the project manager and infer all corresponding mappings from them.

5.2. Rank-dependent state-dependent utility

State-dependent utility concept is evoked when the nature of the state itself is of significance to the decision-maker. One popular example is healthcare, where various outcomes can have major effects on the personal value of the insurance premium (Karni, 1985). In the state-dependent context preferences of the decision maker are given by a binary relation on a set $X = Y^2$. However, unlike in CEU, there exist two (by the number of states) utility functions $U_1 : Y \rightarrow \mathbb{R}$ and $U_2 : Y \rightarrow \mathbb{R}$, so that $x \succsim y \iff C(\nu, (U_1(x_1), U_2(x_2))) \geq C(\nu, (U_1(y_1), U_2(y_2)))$, where x, y are acts, and x_1, x_2, y_1, y_2 are respectively outcomes of x and y in each of the states $(1, 2)$. We note that regions $\{z : z \in X, U_1(z_1) \geq U_2(z_2)\}$ and $\{z : z \in X, U_1(z_1) \leq U_2(z_2)\}$ do not necessarily correspond to comonotonic regions of X anymore. Constant acts also do not have any special value in such context, since $U_1(x_1)$ and $U_2(x_1)$ are not necessarily equal. However, our characterization allows to construct this model. Moreover, as a further generalization, in our framework sets of outcomes in every state can be completely disjoint as well.

5.3. Social choice

If we think of the set N as of a society with two agents, then X is the set of all possible welfare distributions. Moreover, contrary to the classical scenario, agents could be receiving completely different goods, for example X_1 might correspond to healthcare options, whereas X_2 to various educational possibilities. In this case it is not a trivial task to build a correspondence between different options across agents. Our result basically states that, provided the preferences of the social planner abide by the axioms given in Section 3, the decisions are made as if the social planner has associated cardinal utilities with the outcomes of each agent which are *unit* and *level* comparable (cardinal fully comparable or CFC in terms of Roberts (1980)). Such approach is not conventional in social choice problems, where the global (social) ordering is usually not considered as given. Instead, the conditions are normally given on individual utility functions and

the “aggregating” functional that is used to derive the global ordering. However, one of the important questions in social choice literature is that of the interpersonal utility comparability and whether it is justifiable to assume it or not. Our results show that in case the global ordering of alternatives made by the society (or the social planner) satisfy certain conditions, it is in principle possible to have individual preferences represented by utility functions that are not only unit but also level comparable between each other.

Appendix

Subsequent sections are organized as follows. Section A.1 contains a brief sketch of the proof. Section A.3 investigates monotonicity properties, Sections A.2, A.4- A.9 contain the main body of the proof: construction of capacity and value functions, Section A.10 analyses uniqueness of the obtained representation, finally, necessity of the axioms is shown in Section A.11.

A.1. Proof sketch

1. Define extreme points of **SE** and **NW** and temporarily remove them from X (Section A.4).
2. Take any point z , show that there exists an additive representation for \succsim on \mathbf{NW}_z if $z \in \mathbf{NW}$ or \mathbf{SE}_z if otherwise (Section A.4).
3. Having built additive representations for \succsim on \mathbf{SE}_{z_1} and \mathbf{SE}_{z_2} , show that there exists an additive representation on $\mathbf{SE}_{z_1} \cup \mathbf{SE}_{z_2}$ (Section A.4).
4. Cover all **SE** with 3C-cones, and show that the joint representation, call it V^{SE} , can also be extended to cover all **SE** (Section A.4).
5. Perform steps 2 and 3 for **NW** and obtain V^{NW} (Section A.4).
6. Align and scale V^{SE} and V^{NW} such that $V_1^{SE} = V_1^{NW}$ on the common domain, and $V_2^{SE} = \lambda V_2^{NW}$ on their common domain (Section A.5).
7. Pick two points r^0, r^1 from Θ and set r^0 as a common zero. Set $V_1^{SE}(r_1^1) = 1$ and define $\phi_1(x_1) = V^{SE}(x_1), \phi_2(x_2) = V^{SE}(x_2)/V^{SE}(r_2^1)$ (Section A.5).
8. Show that for all $x \in X$ we have $\phi_1(x_1) = \phi_2(x_2)$ iff $x \in \Theta$ unless \succsim can be represented by an additive functional on all X (Section A.6).
9. Representations now are $\phi_1 + k\phi_2$ on **SE** and $\phi_1 + \lambda k\phi_2$ on **NW** (Section A.6).
10. Rescale so that “weights” sum up to one : $\frac{1}{1+k}\phi_1 + \frac{k}{1+k}\phi_2, \frac{1}{1+\lambda k}\phi_1 + \frac{\lambda k}{1+\lambda k}\phi_2$ (Section A.6).
11. Extend the representation to the extreme points (Section A.7).
12. Show that \succsim can be represented on X by these two representations (Section A.8).
13. Show that \succsim can be represented by the Choquet integral (Section A.9).

A.2. Technical lemmas

Lemma 1. *If \succsim satisfies triple cancellation then it is independent.*

Proof. $ap \preccurlyeq ap, aq \succcurlyeq aq, ap \succcurlyeq bp \Rightarrow aq \succcurlyeq bq.$ \square

Lemma 2. $X = \mathbf{SE} \cup \mathbf{NW}.$

Proof. Assume $x \notin \mathbf{SE}, x \notin \mathbf{NW}$. First assume that x is such that its coordinates are not maximal or minimal. Then, there does not exist z such that $x \in \mathbf{SE}_z$ and triple cancellation holds on \mathbf{SE}_z . At the same time, there does not exist z such that $x \in \mathbf{NW}_z$ and triple cancellation holds on \mathbf{NW}_z . This implies that triple cancellation does not hold on \mathbf{SE}_x or \mathbf{NW}_x (otherwise we could have taken $z = x$). This violates **A3**.

Now assume x_1 is maximal. $x \notin \mathbf{SE}$ implies that there exists $y \in \mathbf{SE}_x$ such that triple cancellation holds on \mathbf{NW}_y . But then $x \in \mathbf{NW}$, a contradiction. Other cases are symmetrical. \square

Lemma 3. *Axiom A5 implies the following condition. Let $\{g_1^{(i)} : g_1^{(i)}y_0 \sim g_1^{(i+1)}y_1, g_1^{(i)} \in X_1, i \in N\}$ and $\{h_2^{(i)} : x_0h_2^{(i)} \sim x_1h_2^{(i+1)}, h_2^{(i)} \in X_2, i \in N\}$ be two standard sequences, each entirely contained in \mathbf{NW} or \mathbf{SE} . Assume also, that there exist $z_1, z_2 \in X$, $p, q \in X_2$, $a, b \in X_1$ such that $g_1^{(i)}p, g_1^{(i)}q \in \mathbf{NW}$ or \mathbf{SE} , and $ah_2^{(i)}, bh_2^{(i)} \in \mathbf{NW}$ or \mathbf{SE} for all i , and $g_1^{(i)}p \sim bh_2^{(i)}$ and $g_1^{(i+1)}p \sim bh_2^{(i+1)}$, then for all $i \in N$, $g_1^{(i)}p \sim bh_2^{(i)}$.*

Proof. The proof is very similar to the one from Krantz et al. (1971) (Lemma 5 in section 8.3.1). Assume wlog that $\{g_1^{(i)} : g_1^{(i)}y_0 \sim g_1^{(i+1)}y_1\}$ is an increasing standard sequence on X_1 , which is entirely in \mathbf{SE} , whereas $\{h_2^{(i)} : x_0h_2^{(i)} \sim x_1h_2^{(i+1)}\}$ is an increasing standard sequence on X_2 , and lies entirely in \mathbf{NW} . Assume also for some k it holds $g_1^{(k)}y_0 \sim x_0h_2^{(k)}, g_1^{(k+1)}y_0 \sim x_0h_2^{(k+1)}$. We need to show that $g_1^{(i)}y_0 \sim x_0h_2^{(i)}$ for all i . We will show that $g_1^{(k+2)}y_0 \sim x_0h_2^{(k+2)}$ from which everything holds by induction.

Assume $x_0h_2^{(k+2)} \succ g_1^{(k+2)}y_0$. Since the sequences are increasing, by restricted solvability exists $g \in X_2$ such that $g_1^{(k+2)}y_0 \sim x_0g$. By **A5'**, $g_1^{(k)}y_0 \sim g_1^{(k+1)}y_1, g_1^{(k+1)}y_0 \sim g_1^{(k+2)}y_1, x_0h_2^{(k)} \sim x_1h_2^{(k+1)}$ imply $x_0h_2^{(k+1)} \sim x_1g$. By definition of $\{h_2^{(i)}\}$, $x_0h_2^{(k+1)} \sim x_1h_2^{(k+2)}$. Thus, $x_1h_2^{(k+2)} \sim x_1g$ and by independence $x_0h_2^{(k+2)} \sim x_0g$, hence also $g_1^{(k+2)}y_0 \sim x_0h_2^{(k+2)}$, a contradiction. The case $x_0h_2^{(k+2)} \prec g_1^{(k+2)}y_0$ is symmetrical. Showing that $g_1^{(k-1)}y_0 \sim x_0h_2^{(k-1)}$ can be done in a similar fashion. \square

Lemma 4. *The following statements hold:*

- *If $\mathbf{NW}(\mathbf{SE})$ has only X_2 essential, then for all $x \in \mathbf{NW}(\mathbf{SE})$ there exists $y_2 \in X_2$ such that $x_1y_2 \in \Theta$.*
- *If $\mathbf{NW}(\mathbf{SE})$ has only X_1 essential, then for all $x \in \mathbf{NW}(\mathbf{SE})$ there exists $y_1 \in X_1$ such that $y_1x_2 \in \Theta$.*

Proof. Immediately follows from the structural assumption and closedness of $\mathbf{NW}(\mathbf{SE})$. \square

A.3. Essentiality and monotonicity

In what follows the essentiality of coordinates within various $\mathbf{SE}_z(\mathbf{NW}_z)$ is critical. The central mechanism to guarantee consistency in number of essential coordinates within various 3C-cones is bi-independence which is closely related to comonotonic strong monotonicity of Wakker (1989).

In the Choquet integral representation problem for a heterogeneous product set $X = X_1 \times X_2$, strong monotonicity is actually a necessary condition because of the following. Assume $ap, bp, cp, dp \in \mathbf{SE}$ and $ap \succ bp, cp \sim dp$. Assume also there exist $cq, dq \in \mathbf{NW}$ such that $cq \succ dq$. Then, provided the representation exists, we get

$$\begin{aligned}\alpha_1 f_1(a) + \alpha_2 f_2(p) &> \alpha_1 f_1(b) + \alpha_2 f_2(p) \\ \alpha_1 f_1(c) + \alpha_2 f_2(p) &= \alpha_1 f_1(d) + \alpha_2 f_2(p) \\ \beta_1 f_1(c) + \beta_2 f_2(q) &> \beta_1 f_1(d) + \beta_2 f_2(q).\end{aligned}$$

The first inequality entails $\alpha_1 \neq 0$. From this and the following equality follows $f_1(c) = f_1(d)$, which contradicts with the last inequality. Thus $cq \succ dq$ implies $cp \succ dp$ but only in the presence of $ap \succ bp$ in the same “region” (\mathbf{SE} or \mathbf{NW}). This is also the reason behind the name we gave to this condition - “bi-independence”.

Lemma 5. *Pointwise monotonicity.*

If for all $i, j \in N$ we have $a_i x_j \succcurlyeq a_i y_j$ for all $a_i \in X_i$, then $x \succcurlyeq y$.

Proof. $x = x_1 x_2 \succcurlyeq x_1 y_2 \succcurlyeq y_1 y_2 = y$. □

Bi-independence, together with the structural assumption also implies some sort of “strong monotonicity”.

Lemma 6. *If X_1 is essential on $\mathbf{SE}(\mathbf{NW})$, $a \succcurlyeq_1 b$ iff $ap \succ bp$ for all $ap, bp \in \mathbf{NW}$.*

Proof. Let X_1 be essential on \mathbf{SE} . By the structural assumption, $a \succcurlyeq_1 b$ implies existence of some $q \in X_2$ such that $aq \succ bq$. If $aq, bq \in \mathbf{SE}$ the result follows by independence. If $aq, bq \in \mathbf{NW}$ the result follows by bi-independence. If $bq \in \mathbf{NW}, aq \in \mathbf{SE}$, then by closedness assumption there exists $cq \in \Theta$, and either $bq \succ cq$, or $cq \succ aq$. The result follows by independence or bi-independence. □

Conceptually, Lemma 6 implies that if a coordinate is essential on some 3C-cone $\mathbf{NW}_z(\mathbf{SE}_z)$, then it is also essential on $\mathbf{NW}_x(\mathbf{SE}_x)$ for all $x \in \mathbf{NW}(\mathbf{SE})$. This allows us to make statements like “coordinate i is essential on \mathbf{NW} ”.

A.4. Building additive value functions on NW and SE

In this section we assume that $\mathbf{SE}(\mathbf{NW})$ has two essential coordinates.

A.4.1. Covering \mathbf{SE} and \mathbf{NW} with maximal \mathbf{SE}_z and \mathbf{NW}_z

In the sequel we could have covered areas \mathbf{SE} and \mathbf{NW} by sets $\mathbf{SE}_z(\mathbf{NW}_z)$ for all $z \in \mathbf{SE}(\mathbf{NW})$, but it is convenient to introduce the following lemma.

Lemma 7. *For every $x \in \mathbf{SE}$ there exists $z \in \Theta$ such that $\mathbf{SE}_x \subset \mathbf{SE}_z$. Accordingly, for every $y \in \mathbf{NW}$ there exists $z \in \Theta$ such that $\mathbf{NW}_y \subset \mathbf{NW}_z$.*

Proof. Take $x \in \mathbf{SE}$ such that $x \notin \mathbf{NW}$. If there exists $y \in \mathbf{NW}_x$ such that $y \in \mathbf{NW}$, then by closedness assumption there must exist either $ay_2 \in \Theta$ and $x \in \mathbf{SE}_{ay_2}$ or $x_1p \in \Theta$ and $x \in \mathbf{SE}_{x_1p}$. If such y does not exist, $X = \Theta$. Other cases are symmetrical. \square

It follows from Lemma 7 that $\mathbf{SE} = \bigcup_{z \in \Theta} \mathbf{SE}_z$, while $\mathbf{NW} = \bigcup_{z \in \Theta} \mathbf{NW}_z$. Comparing this to definitions of \mathbf{SE} and \mathbf{NW} we are able to define also the following notions:

Definition 10. *We write $x \in \mathbf{SE}_{ext}$ and say that $x \in X$ is extreme in \mathbf{SE} if $x \in \Theta$ and $[x_2 \text{ is minimal or } x_1 \text{ is maximal}]$. We write $x \in \mathbf{NW}_{ext}$ and say that $x \in X$ is extreme in \mathbf{NW} if $x \in \Theta$ and $[x_1 \text{ is minimal or } x_2 \text{ is maximal}]$. $x \in X$ is extreme if it is extreme in \mathbf{SE} or in \mathbf{NW} .*

Note that contrary to the homogeneous case $X = Y^n$, extreme points for \mathbf{SE} and \mathbf{NW} can be asymmetric, i.e. if a point z is extreme in \mathbf{SE} it is not necessarily extreme in \mathbf{NW} .

A.4.2. Representations within \mathbf{SE}_z

In the following we will build an additive representation on \mathbf{SE} . The case of \mathbf{NW} is symmetric. We proceed by building representations on sets \mathbf{SE}_z for all $z \in \Theta \setminus \mathbf{SE}_{ext}$ (i.e. for all non-extreme points of Θ).

Essential coordinates.. For now we assume that both coordinates are essential on \mathbf{NW} and \mathbf{SE} .

Theorem 2. *For any $z \in \Theta \setminus \mathbf{SE}_{ext}$ there exists an additive representation of \succsim on \mathbf{SE}_z :*

$$x \succsim y \Leftrightarrow V_1^z(x_1) + V_2^z(x_2) \geq V_1^z(y_1) + V_2^z(y_2).$$

Proof. \mathbf{SE}_z is a Cartesian product, \succsim is a weak order on \mathbf{SE}_z , \succsim satisfies triple cancellation on \mathbf{SE}_z , \succsim satisfies Archimedean axiom on \mathbf{SE}_z , both coordinates are essential. It remains to show that \succsim satisfies restricted solvability on \mathbf{SE}_z .

Assume that for some $xa, y, xc \in \mathbf{SE}_z$, we have $xa \succsim y \succsim xc$, hence exists $b \in X_2 : xb \sim y$. We need to show that $xb \in \mathbf{SE}_z$. If $xb \sim xa$ or $xb \sim xc$, then the result is immediate. Hence, assume $xa \succ xb \succ xc$. By definition, $x \in \mathbf{SE}_z$, if $x \succ_1 z_1$, and $z_2 \succ_2 b$. For xb we need to check only the latter condition. It holds, since $xa \succ xb \succ xc$, and by weak separability $a \succ_2 b$.

Therefore all conditions for the existence of an additive representation are met (Krantz et al., 1971). \square

A.4.3. Joining representations for different \mathbf{SE}_z (or \mathbf{NW}_z)

This section closely follows (Wakker, 1991a).

Theorem 3. *There exists an additive interval scale V^{SE} on $\bigcup \mathbf{SE}_z$, with $z \in \Theta \setminus \mathbf{SE}_{ext}$, which represents \succsim on every \mathbf{SE}_z .*

Proof. Choose the “reference” points - pick any $r \in \mathbf{SE}$ and any $r^0, r^1 \in \mathbf{SE}_r$ such that $r_1^1 p \succsim r_1^0 p$ for every $p \in X_2$. Set $V_1^r(r_1^0) = 0, V_2^r(r_2^0) = 0, V_1^r(r_1^1) = 1$. Now, we align representations on the other sets \mathbf{SE}_z with the reference one. Assume that for some $z \in \Theta$ we have already obtained an additive representation V^z on \mathbf{SE}_z . Observe that V^z and V^r are additive value functions for \succsim on $\mathbf{SE}_z \cap \mathbf{SE}_r$. Moreover $\mathbf{SE}_z \cap \mathbf{SE}_r = \mathbf{SE}_{q_1}$, where $q_1 = r_1$ if $r_1 \succsim_1 z_1$ and $q_1 = z_1$ if the opposite is true. Similarly, $q_2 = r_2$ if $az_2 \succsim ar_2$ for all $a \in X_1$ and $q_2 = z_2$ in the opposite case. Hence, uniqueness results from Krantz et al. (1971) can be applied. In particular, this means that on $\mathbf{SE}_z \cap \mathbf{SE}_r$ we have $V_i^r = \alpha V_i^z + \beta_i$, so the functions are defined up to a common unit and location.

We choose the unit and location of V_1^z so that $V_1^z(x_1) = V_1^r(x_1)$ for all $x \in \mathbf{SE}_z \cap \mathbf{SE}_r$. Next, we choose the location of V_2^z so that it coincides with V_2^r on $\mathbf{SE}_z \cap \mathbf{SE}_r$.

Finally, we show that $V_i^s(x_i) = V_i^t(x_i)$ for any $s, t \in \Theta$ and $x \in \mathbf{SE}_s \cap \mathbf{SE}_t$. This immediately follows, since V^s and V^t coincide (with V^r) on $\mathbf{SE}_s \cap \mathbf{SE}_t \cap \mathbf{SE}_r$. This defines their unit and locations, hence they also coincide on $\mathbf{SE}_s \cap \mathbf{SE}_t$. Now define V^{SE} as a function which coincides with V^{z_i} on the respective domains \mathbf{SE}_{z_i} . By the above argument, this function is well-defined. \square

Theorem 4. *Representation V^{SE} obtained in Theorem 3 is globally representing on $\mathbf{SE} \setminus \mathbf{SE}_{ext} = \bigcup_{z \in \Theta \setminus \mathbf{SE}_{ext}} \mathbf{SE}_z$.*

Proof. Let $x \succsim y$. There can be two cases. First, assume that $x_2 \succsim_2 y_2$, but $y_1 \succsim_1 x_1$ (or vice versa). In this case, x and y belong to the same \mathbf{SE}_z (e.g. \mathbf{SE}_x) and therefore V^{SE} is a valid representation.

Next, assume that $x_j \succsim_j y_j$ for all $i, j \in N$. Assume, that $x \in \mathbf{SE}_s, y \in \mathbf{SE}_t$. Observe that $x_1 y_2 \in \mathbf{SE}_s \cap \mathbf{SE}_t$ because by the made assumptions, $x_1 y_2 \in \mathbf{SE}_x, x_1 y_2 \in \mathbf{SE}_y$. By pointwise monotonicity (Lemma 5) $x_1 x_2 \succsim x_1 y_2 \succsim y_1 y_2$, hence $V_1(x_1) + V_2(x_2) \geq V_1(x_1) + V_2(y_2) \geq V_1(y_1) + V_2(y_2)$, with first inequalities lying in \mathbf{SE}_s , and second in \mathbf{SE}_t . The reverse implication is also true. \square

A.5. Aligning V^{SE} and V^{NW}

First we will show that it is not possible for the common domain of V_i^{SE} and V_i^{NW} for some i to contain a single point.

A.5.1. Analysis of the common domain of V^{SE} and V^{NW}

Lemma 8. *Let $a_0 \succsim_1 b_0$, and for some $p \in X_2$ we have $a_0 p, b_0 p \in \Theta$. Define $X_{a_0 b_0} = \{x_1 : x_1 \in X_1, b_0 \succsim_1 x_1 \succsim_1 a_0\}$. Then, triple cancellation holds everywhere on $X_{a_0 b_0} \times X_2$.*

Proof. All points in the below proof are from $X_{a_0 b_0} \times X_2$. Let $ax \precsim by, aw \succsim bz, cx \succsim dy$. We will show that together with the assumptions of the Lemma, this implies $cw \succsim dz$.

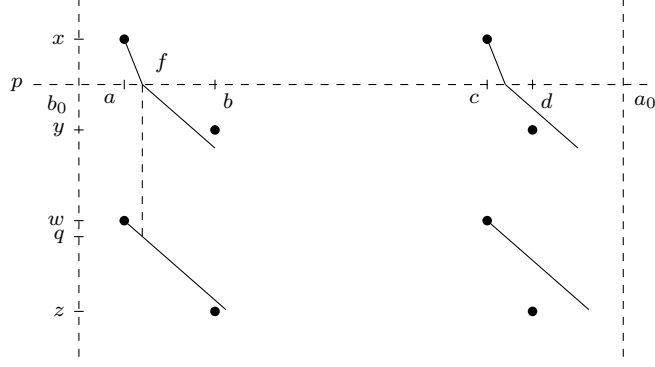


Figure 1: Lemma 8

The case when all points belong to **SE** or **NW**, or two pairs belong to **SE** and two to **NW** is covered by **A4**. Thus, assume wlog $x \succ_2 p$, so that $ax, cx \in \mathbf{NW}$ and the remaining points are in **SE** (Fig. 1). Assume also $dp \succ cp$ and $b \succ_1 a$. Assume also $ax \succ ap$ (hence by independence also $cx \succ cp$), $bp \succ by$ (hence also $dp \succ dy$), otherwise the result immediately follows by **A4** (e.g. if $ax \sim ap$, we can replace ax by ap and cx by cp in the assumptions of the lemma, which brings all points to **SE**).

1. $ax \preccurlyeq by$.
 $bp \succ by$, hence $ax \prec bp$. $ax \succ ap$, hence $bp \succ by \succcurlyeq ax \succ ap$, $bp \succ ap$, therefore, by restricted solvability exists $fp \sim ax$. Also, $fp \succ ap, bp \succ fp$.
2. $cx \succ dy$. There can be two cases:
 - a) If $cx \preccurlyeq dp$, then $dp \succcurlyeq cx \succ cp$, hence exists $gp \sim cx$.
 - b) $cx \succ dp$.
3. $aw \succ bz$.
Solve for q : $aw \sim fq$. By the results in point 1 and independence we have $fw \succ aw \succcurlyeq bz \succ fz$, therefore by restricted solvability exists $q : fq \sim aw$.
4. Cases correspond to those in point 2 above:
 - a) $fp \sim ax, gp \sim cx, aw \sim fq$, hence by **A4** $cw \sim gq$
 $fp \preccurlyeq by, gp \succcurlyeq dy, fq \succcurlyeq bz$, hence by **A4** $gq \succcurlyeq dz$ and $cw \succcurlyeq dz$.
 - b) $ax \prec bp, cx \succ dp, aw \succcurlyeq bz$, hence by **A4** $cw \succcurlyeq dz$.

□

From this it follows that it is impossible that for some i the common domain of V_i^{SE} and V_i^{NW} includes a single point. Let (wlog) $i = 1$ and $a \in X_1$ be such a point. Apparently $ap \in \Theta$ for all $a \in X_1$. Then, from Lemma 8 it follows that **SE** = **NW** = X .

A.5.2. Aligning representations on **SE** and **NW**

There can be four cases, depending on the number of essential coordinates on **NW** and **SE**:

1. Both areas have two essential coordinates;
2. One area has two essential coordinate, another has one essential coordinate;
3. Both areas have one essential coordinates;
4. An area does not have any essential coordinates.

We start with the case where both coordinates are essential on **NW** and **SE**.

A.5.2.1. Both coordinates are essential

Lemma 9. Choose $r_1^0 \in X_1$ and $r_1^1 \in X_1$ from the common domain of V_1^{SE} and V_1^{NW} such that $r_1^1 \succ_1 r_1^0$, and set $V_1^{SE}(r_1^0) = V_1^{NW}(r_1^0) = 0$ and $V_1^{SE}(r_1^1) = V_1^{NW}(r_1^1) = 1$. Then, $V_1^{SE}(x) = V_1^{NW}(x)$ on all x from their common domain.

Proof. This follows directly from **A4**. Assume, we want to show that $V_1^{SE}(y_1) = V_1^{NW}(y_1)$ for some $y_1 \in X_1$. Starting from r_1^0 build any standard sequence on X_1 in **SE**, say $\{\alpha_1^{(i)} : \alpha_1^{(i)} y_1^s \sim \alpha_1^{(i+1)} y_2^s\}$. Then, all $\alpha_1^{(i)} y_1^n, \alpha_1^{(i)} y_2^n$ which are in **NW** also form a sequence in **NW**: if $\alpha_1^{(i)} y_1^s \sim \alpha_1^{(i+1)} y_2^s, \alpha_1^{(i+1)} y_1^s \sim \alpha_1^{(i+2)} y_2^s$ and $\alpha_1^{(i)} y_1^n \sim \alpha_1^{(i+1)} y_2^n$, for some $y_1^n, y_2^n \in X_2$, then by **A4**, necessarily $\alpha_1^{(i+1)} y_1^n \sim \alpha_1^{(i+2)} y_2^n$.

Now let

$$\begin{aligned} 1 = V_1^{SE}(r_1^1) &\approx n[V_2^{SE}(y_2^s) - V_2^{SE}(y_1^s)] \\ V_1^{SE}(y_1) &\approx m[V_2^{SE}(y_2^s) - V_2^{SE}(y_1^s)] \approx \frac{m}{n}. \end{aligned}$$

Such n and m exist by the Archimedean axiom. By the argument above we get

$$\begin{aligned} 1 = V_1^{NW}(r_1^1) &\approx n[V_2^{NW}(y_2^n) - V_2^{NW}(y_1^n)] \\ V_1^{NW}(y_1) &\approx m[V_2^{NW}(y_2^n) - V_2^{NW}(y_1^n)] \approx \frac{m}{n} \end{aligned}$$

By denserangedness, approximations become exact in the limit, so we obtain $V_1^{SE}(y_1) = V_1^{NW}(y_1)$ on all $y_1 \in X_1$ from their common domain. \square

Lemma 10. Assume, V^{SE} is an additive representation of \succsim on $\mathbf{SE} \setminus \mathbf{SE}_{ext}$, and V^{NW} is a representation on $\mathbf{NW} \setminus \mathbf{NW}_{ext}$, with V_1^{SE} and V_1^{NW} scaled so that they have a common zero and unit (as in Lemma 9). Then, $V_2^{SE} = \lambda V_2^{NW}$ on the common domain.

Proof. By Lemma 9, $V_1^{SE} = V_1^{NW}$ on the common domain. Assume $V_2^{SE}(r_2^2) = \lambda, V_2^{NW}(r_2^2) = 1$. We will now show that $V_2^{SE}(x_2) = \lambda V_2^{NW}(x_2)$ for all $x_2 \in X_2$ from the common domain of these functions. Construct a standard sequence within \mathbf{SE}_z , this time on X_2 . By **A4**, it is also a sequence in **NW**. We obtain

$$\begin{aligned} \lambda = V_2^{SE}(r_2^2) &\approx n[V_1^{SE}(x_1^s) - V_1^{SE}(x_1^s)] \\ V_2^{SE}(x_2) &\approx m[V_1^{SE}(x_1^s) - V_1^{SE}(x_1^s)] \approx \frac{\lambda m}{n} \end{aligned}$$

By the argument above we get

$$\begin{aligned} 1 = V_2^{NW}(r_2^2) &\approx n[V_2^{NW}(x_1^n) - V_2^{NW}(x_1^n)] \\ V_2^{NW}(x_2) &\approx m[V_1^{NW}(x_1^n) - V_1^{NW}(x_1^n)] \approx \frac{m}{n} \end{aligned}$$

From this in the limit we obtain $V_2^{SE}(x_2) = \lambda V_2^{NW}(x_2)$ on all $x_2 \in X_2$ from the common domain of V_2^{SE} and V_2^{NW} . \square

At this point we can drop superscripts and say that we have representations $V_1 + V_2$ on **SE** and $V_1 + \lambda V_2$ on **NW**. Fix two non-extreme points in Θ : r^0 and r^1 , such that $r_1^1 \succ_1 r_1^0$ and $r_2^1 \succ_2 r_2^0$. If such points do not exist, then by Lemma 8 triple cancellation holds everywhere and \succ can be represented by an additive function (i.e. $\lambda = 1$). Rescale V_1 and V_2 so that $V_1(r_1^0) = 0$, $V_2(r_2^0) = 0$, $V_1(r_1^1) = 1$. Assume that after rescaling we get $V_1(r_2^1) = k$. Define $\phi_2(x_2) = V_2(x_2)/k$, i.e. $\phi_2(r_2^1) = 1$. Define $\phi_1(x_1) = V_1(x_1)$. Thus we get representations $\phi_1 + k\phi_2$ on **SE** and $\phi_1 + \lambda k\phi_2$ on **NW**. Finally rescale in the following way: $\frac{1}{1+k}\phi_1 + \frac{k}{1+k}\phi_2$ on **SE** and $\frac{1}{1+\lambda k}\phi_1 + \frac{\lambda k}{1+\lambda k}\phi_2$ on **NW**. We have thus defined the following representations:

$$\begin{aligned} \phi^{SE}(x) &= \frac{1}{1+k}\phi_1(x_1) + \frac{k}{1+k}\phi_2(x_2) \\ \phi^{NW}(x) &= \frac{1}{1+\lambda k}\phi_1(x_1) + \frac{\lambda k}{1+\lambda k}\phi_2(x_2). \end{aligned} \tag{2}$$

Note, that it follows that $\phi^{SE}(r^1) = \phi^{NW}(r^1) = 1$.

A.5.2.2. One area has a single essential coordinate

Assume **SE** has two essential coordinates and **NW** only has X_1 essential. After an additive representation V^{SE} has been built on **SE**, and re-scaled as in (2) we have values ϕ_1 and ϕ_2 for all points in **SE**, in particular those in Θ . Let $\phi^{NW}(x) = \phi_1(x_1) + 0\phi_2(x_2)$ (in other words, set $\lambda = 0$ in (2)) for those x_i where ϕ_i are defined. By structural assumption, bi-independence and additivity ϕ^{NW} represents \succ on those points for which it is defined. For example, let $ap, bp \in \mathbf{NW}$ be such that $ap \succ bp$. Since both coordinates are essential on **SE** by bi-independence we get also $aq \succ bq$ for all $q \in X_2$ such that $aq, bq \in \mathbf{SE}$. Additivity implies $\phi_1(a) > \phi_1(b)$. For the remaining $x_1 \in X_1$, i.e. for $x_1 \in X_1$ such that there are no points in Θ first coordinate of which is x_1 , build a simple ordinal representation. Lemma 4 shows that values for all $x_2 \in X_2$ have already been defined at this point. Other cases are similar.

A.5.2.3. Both areas have a single essential coordinate

An interesting result is that **A3** is sufficient for characterization of cases where both **SE** and **NW** have one essential coordinate. There are two cases in total:

1. X_1 is essential on **NW**, X_2 is essential on **SE**;
2. X_2 is essential on **NW**, X_1 is essential on **SE**.

We will need the following lemma.

Lemma 11. *Let X_1 be essential on **NW** and X_2 be essential on **SE** or X_2 be essential on **NW** and X_1 be essential on **SE**. Then, either for all $x \in \mathbf{SE}$ exists $z \in \Theta$ such that $z \sim x$, or for all $y \in \mathbf{NW}$ exists $z \in \Theta$ such that $z \sim y$.*

Proof. We only consider one case, others being symmetrical. Assume X_1 is essential on **SE** and X_2 is essential on **NW**. Assume also there exists $x \in \mathbf{SE}$ such that $x \succ z$ for all $z \in \Theta$, in particular some maximal z^{\max} . We will show that this implies that there does not exist $y \in \mathbf{NW}$ such that $y \succ z$ or $z \succ y$ for all $z \in \Theta$.

Assume such y exists. Take $x_1 y_2$. By **A3** it belongs either to **SE** or **NW**. If it belongs to **NW**, by closedness assumption exists $x_1 t \in \Theta$. We get $x_1 t \sim x \succ z^{\max}$ - a contradiction. If $x_1 y_2 \in \mathbf{SE}$, exists $ay_2 \in \Theta$. We have $y \sim ay_2$ which contradicts both $y \succ z$ and $z \succ y$ for all $z \in \Theta$.

Finally, we need to show that Θ does not have gaps. Assume there exists $y \in \mathbf{NW}$ and $z^1, z^2 \in \Theta$ such that $z^1 \succ y \succ z^2$ but there is no $z \in \Theta$ such that $z \sim y$. Since only X_2 is essential on **NW**, we get $z_1^1 y_2 \in \mathbf{SE}$. By closedness assumption exists $x_1 \in X_1$ such that $x_1 y_2 \in \Theta$. Since only X_2 is essential we conclude $x_1 y_2 \sim y$, which is a contradiction. Therefore, for every $y \in \mathbf{NW}$ there exists $z \in \Theta$ such that $y \sim z$. \square

Defining value functions.. Lemma 11 guarantees that for all points in **SE** or all points in **NW** exists an equivalent point in Θ . Assume for example that Θ is such that for all $x \in \mathbf{SE}$ exists $z \in \Theta$. Assume also that X_1 is essential on **SE** and X_2 is essential on **NW**. Now define value functions $\phi_1 : X_1 \rightarrow \mathbb{R}$ and $\phi_2 : X_2 \rightarrow \mathbb{R}$ as follows. Choose ϕ_1 to be any real-valued function such that $\phi_1(x_1) > \phi_1(y_1)$ iff $x_1 \succ_1 y_1$. Now for all z from Θ set $\phi_2(z_2) = \phi_1(z_1)$. Finally, extend ϕ_2 to the whole X_2 by choosing any function such that $\phi_2(x_2) > \phi_2(y_2)$ iff $x_2 \succ_2 y_2$. Lemma 4 guarantees that the functions have been defined for all $x_1 \in X_1$ and all $x_2 \in X_2$.

A.5.2.4. Areas without essential coordinates

Lemma 12. *If **A1** - **A9** and the structural assumption hold, there can not be $\mathbf{NW}_z(\mathbf{SE}_z)$ with no essential coordinates.*

Proof. Assume for some $z \in \Theta$ the set \mathbf{NW}_z has no essential coordinates. By bi-independence and the structural assumption it follows that there are no essential coordinates on any \mathbf{NW}_z . This implies (by **A7**) that both coordinates are essential on **SE**. Take $ap, bp \in \mathbf{NW}_z$. Apparently, $ap \sim bp$. By structural assumption there must exist $q \in X_2$ such that $aq \succ bq$. It can't be that $aq, bq \in \mathbf{NW}$, hence $aq, bq \in \mathbf{SE}$.

By closedness assumption there exist $w, z \in X_2$ such that $aw, bz \in \Theta$. Also, since no coordinate is essential in **NW** we have $aw \sim bz$. Since $aq \succ bq$ it must be $bz \succ bw$, since otherwise by strict monotonicity (Lemma 6) it can't be that $aw \sim bz$.

By independence we have $aw \succ bw$. By definition of **NW** and **SE** we have $aw \in \mathbf{SE}$ (since by weak separability $a \succ_1 b$) and $aw \in \mathbf{NW}$ (since by weak separability $z \succ_2 w$). Hence, by independence it must be $az \succ aw$ (since $az \in \mathbf{SE}$) and $az \sim aw$ (since $az \in \mathbf{NW}$). We have arrived at a contradiction. \square

A.6. Properties of the intersection of SE and NW

Lemma 13. *For any non-extreme $x \in X$ we have:*

$$x \in \Theta \Rightarrow \phi_1(x_1) = \phi_2(x_2),$$

unless \succsim can be represented by an additive function (i.e $\lambda = 1$ in (2)).

For the case when both **NW** and **SE** have a single essential coordinate the result holds by definition of ϕ_i , so for the remainder of this section we assume that **SE** or **NW** has two essential coordinates.

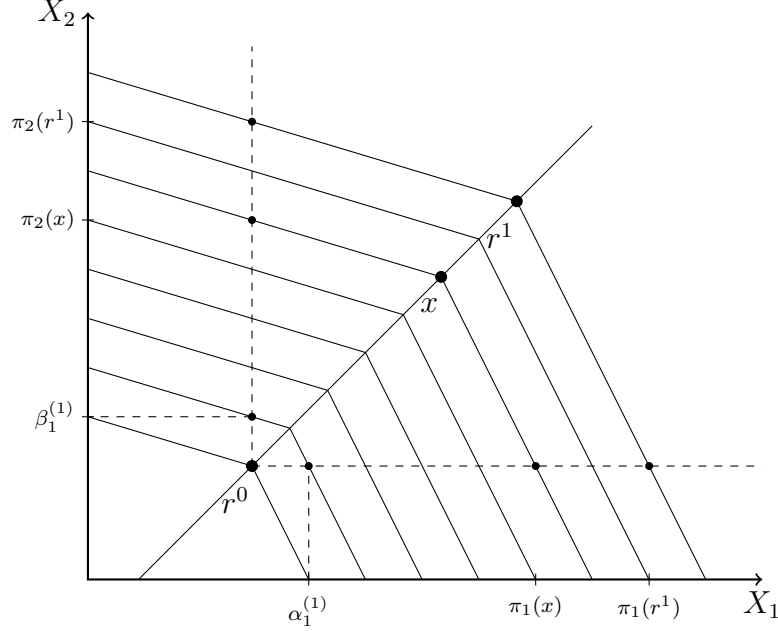


Figure 2: Lemma 13

Proof. We start with a case where both coordinates are essential on **SE** and **NW**. Assume also $x \succsim r^0$ (without loss of generality, other cases are symmetrical and can be proved by the same technique). We are going to show that $x \in \Theta \Rightarrow \phi_1(x_1) = \phi_2(x_2)$ or $\lambda = 1$.

Assume for now that we can find the following solutions:

- Solve for $\pi_1(r^1) : \pi_1(r^1)r_2^0 \sim r^1$.
- Solve for $\pi_2(r^1) : r_1^0\pi_2(r^1) \sim r^1$.
- Solve for $\pi_1(x) : \pi_1(x)r_2^0 \sim x$.
- Solve for $\pi_2(x) : r_1^0\pi_2(x) \sim x$.

Pick $\alpha_1^{(1)} \in X_1$ such that $r_1^1 \succsim_1 \alpha_1^{(1)}$, it exists by denserangedness. Solve for $\beta_2^{(1)} : \alpha_1^{(1)}r_2^0 \sim r_1^0\beta_2^{(1)}$, exists by restricted solvability.

Now build an increasing standard sequence $\alpha_1^{(i)} : \alpha_1^{(0)} = r_1^0$ on X_1 which lies in **SE** and an increasing standard sequence $\beta_2^{(i)} : \beta_2^{(0)} = r_2^0$ on X_2 which lies in **NW** (see Fig. 2). Since $\pi_1(r^1)r_2^0 \sim r_1^0\pi_2(r^1)$, by **A5** (Lemma 3) we have for some m :

$$\alpha_1^{(m-1)}r_2^0 \succsim \pi_1(r^1)r_2^0 \succsim \alpha_1^{(m)}r_2^0 \iff r_1^0\beta_2^{(m-1)} \succsim r_1^0\pi_2(r^1) \succsim r_1^0\beta_2^{(m)}.$$

From this (and since $\phi_1(r_1^0) = \phi_2(r_2^0) = 0$) it follows that $\phi_1(\pi_1(r^1)) \approx m\phi_1(\alpha_1^{(1)})$, $\phi_2(\pi_2(r^1)) \approx m\phi_2(\beta_2^{(1)})$ and, since $\phi^{SE}(\pi_1(r^1)r_2^0) = \phi^{SE}(r^1) = \phi^{NW}(r^1) = \phi^{NW}(r_1^0\pi_2(r^1))$, we obtain:

$$\frac{1}{1+k}m\phi_1(\alpha_1^{(1)}) = \frac{\lambda k}{1+\lambda k}m\phi_2(\beta_2^{(1)}). \quad (3)$$

Similarly, $\pi_1(x)r_2^0 \sim r_1^0\pi_2(x)$, so by **A5** (Lemma 3) we have

$$\alpha_1^{(n-1)}r_2^0 \succsim \pi_1(x)r_2^0 \succsim \alpha_1^{(n)}r_2^0 \iff r_1^0\beta_2^{(n-1)} \succsim r_1^0\pi_2(x) \succsim r_1^0\beta_2^{(n)}.$$

From this follows that $\phi_1(\pi_1(x)) \approx n\phi_1(\alpha_1^{(1)})$, $\phi_2(\pi_2(x)) \approx n\phi_2(\beta_2^{(1)})$ and by (3) it follows that $\phi^{SE}(\pi_1(x)r_2^0) = \phi^{NW}(r_1^0\pi_2(x))$. Hence

$$\frac{1}{1+k}\phi_1(x_1) + \frac{k}{1+k}\phi_2(x_2) = \frac{1}{1+\lambda k}\phi_1(x_1) + \frac{\lambda k}{1+\lambda k}\phi_2(x_2),$$

and so $\phi_1(x_1) = \phi_2(x_2)$ or $\lambda = 1$ (i.e. the structure is additive).

We need to revisit the case where solutions mentioned in the beginning do not exist. Consider Figure 3. Assume this time that there does not exist $\pi_1(r^1)$ such that $r^1 \sim \pi_1(r^1)r_2^0$. If we choose the step in the standard sequence $\alpha_1^{(i)}$ small enough so that there exist $\alpha_1^{(k+1)}, \alpha_1^{(k+2)}$ such that $\alpha_1^{(k+1)}r_2^0 \not\succsim r_1^0r_2^0$ and $\alpha_2^{(k+2)}r_2^0 \succsim r_1^0r_2^0$ (which we can do by non-maximality of r^1 and denserangedness of \succsim), then we can “switch” from the standard sequence $\alpha_1^{(i)}$ on X_1 to the standard sequence $\gamma_2^{(i)}$ on X_2 keeping the same increment in value between subsequent members of the sequence. Indeed, $\gamma_2^{(k)} : r_1^1\gamma_2^{(k)} \sim \alpha_1^{(k)}r_2^0$ and $\gamma_2^{(k+1)} : r_1^1\gamma_2^{(k+1)} \sim \alpha_1^{(k+1)}r_2^0$ exist by monotonicity and restricted solvability, so does $x_1 : x_1\gamma_2^{(k)} \sim r_1^1\gamma_2^{(k+1)} \sim \alpha_1^{(k+1)}r_2^0$, and by **A5** (Lemma 3) we get $[r_1^1\gamma_2^{(k)} \sim r_1^0\beta_2^{(k)}, r_1^1\gamma_2^{(k+1)} \sim r_1^0\beta_2^{(k+1)}] \Rightarrow \gamma_2^{(i)} \sim \beta_2^{(i)}$ for all i . Note that $\gamma_2^{(i)}r_2^1$ are in **SE** for all i such that $r^1 \succsim \gamma_2^{(i)}$ since r^0 and r^1 are in **SE**. By monotonicity and restricted solvability there exists i such that $x_1r_2^1 \succsim x_1\gamma_2^{(i+1)} \succsim r^1 \succsim x_1\gamma_2^{(i)}$. Finally, the increment in value is the same between members of α_i and $\gamma_2^{(i)}$ since $\alpha_k \sim \gamma_2^{(k)}$ and $\alpha_{k+1} \sim \gamma_2^{(k+1)}$. The result then follows as above.

Finally, we look at the case where only one coordinate is essential on either **NW** or **SE**. First assume that X_2 is essential on **NW**. We defined $\phi^{NW}(x) = 0\phi_1(x_1) + \phi_2(x_2)$. Definition implies $\phi^{NW}(r^0) = 0, \phi^{NW}(r^1) = 1$. Build a standard sequence $\{\alpha_1^{(i)}\}$ on X_1 from r^0 to r^1 (in case there exists a solution for $r^1 \sim \pi_1(r^1)r_2^0$, otherwise use the approach detailed in the previous paragraph), setting $\alpha_1^{(0)} = r_1^0$. Take $\alpha_1^{(1)}r_2^0$ and $\alpha_1^{(2)}r_2^0$. By restricted solvability there must exist $\beta_2^{(1)}$ and $\beta_2^{(2)}$, such that $\alpha_1^{(1)}r_2^0 \sim r_1^0\beta_2^{(1)}$ and $\alpha_1^{(2)}r_2^0 \sim r_1^0\beta_2^{(2)}$. By closedness assumption for $\beta_2^{(1)}, \beta_2^{(2)}$ there must exist x_1, x_2 such that $x_1\beta_2^{(1)} \in \Theta, x_2\beta_2^{(2)} \in \Theta$. Also, since only X_2 is essential, we get $x_1\beta_2^{(1)} \sim r_1^0\beta_2^{(1)}, x_2\beta_2^{(2)} \sim r_1^0\beta_2^{(2)}$. By weak monotonicity and definition of **SE**, $\alpha_1^{(2)}\beta_2^{(1)} \succsim \alpha_1^{(2)}r_2^0 \sim x_2\beta_2^{(2)} \succsim x_2\beta_2^{(1)}$, hence by restricted solvability exists $z_1 : x_2\beta_2^{(2)} \sim z_1\beta_2^{(1)}$. By **A5** then $x_2\beta_2^{(1)} \sim z_1r_2^0$. By additivity $x_2\beta_2^{(2)} \sim z_1\beta_2^{(1)}$ and $x_2\beta_2^{(1)} \sim z_1r_2^0$ entail $\phi_2(b_2) - \phi_2(b_1) = \phi_2(b_1) - \phi_2(r_2^0)$. From this the result follows as in the proof above. If now X_1 is essential on **NW** repeat the proof as above this time starting the sequence from r^1 “towards” r^0 .

□

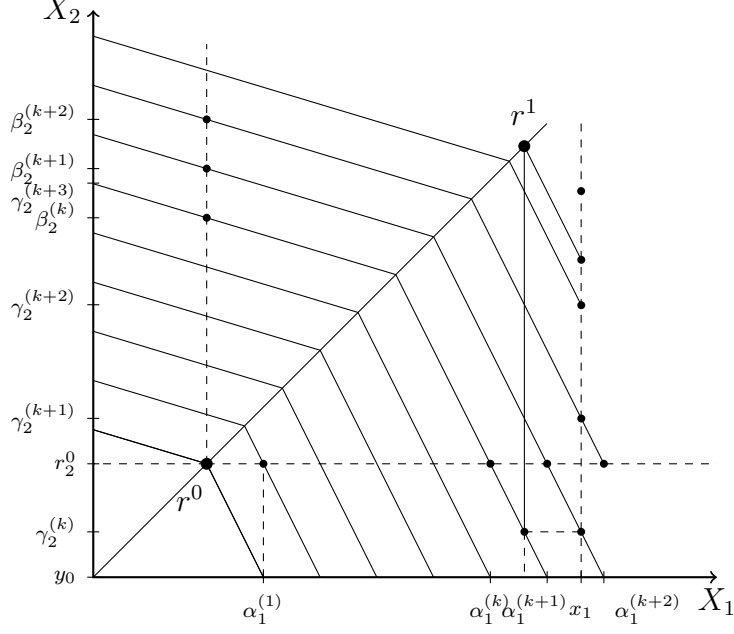


Figure 3: Lemma 13 - changing direction

Lemma 14. *The following statements hold or \succsim has an additive representation:*

1. *If $ap \in \Theta$ then for no $b \in X_1$ holds $bp \in \Theta$ and also for no $q \in X_2$ holds $aq \in \Theta$.*
2. *$x \in \mathbf{SE} \Rightarrow \phi_1(x_1) \geq \phi_2(x_2)$, $y \in \mathbf{NW} \Rightarrow \phi_2(y_2) \geq \phi_1(y_1)$.*

Proof. 1. Assume $ap, bp \in \Theta$. Assume first, that **SE** or **NW** has two essential coordinates. If $ap \sim bp$, then $aq \sim bq$ for all $q \in X_2$, which violates the structural assumption. Let $ap \succ bp$. By denserangedness there exists also cp such that $ap \succ cp \succ bp$. If p is minimal, then by definition of **SE**, $bp \in \mathbf{SE}$ if for no $c \in X_1$ such that $c \succ_1 b$ holds $cp \in \mathbf{NW}$. This is obviously violated by ap . If p is maximal, then by definition of **NW**, $ap \in \mathbf{NW}$ if for no $c \in X_1$ such that $a \succ_1 c$ holds $cp \in \mathbf{SE}$, which is violated by bp . Hence, p is not maximal or minimal. If a is maximal, then cp, bp are non-extreme, hence by Lemma 13, $\phi_1(c) = \phi_2(p) = \phi_1(b)$, hence $bq \sim cq$ for all $q \in X_2$. If b is minimal, ap, bp are non-extreme, hence $\phi_1(a) = \phi_2(p) = \phi_1(c)$, hence $aq \sim cq$ for all $q \in X_2$. In both cases structural assumption is violated.

2. Pick any $bq \in \mathbf{SE}$. By Lemma 7 there exists $ap \in \Theta$ such that $bq \in \mathbf{SE}_{ap}$, hence $b \succ_1 a$ and $p \succ_2 q$. By Lemma 13 $\phi_1(a) = \phi_2(p)$. We also have $\phi_1(b) \geq \phi_1(a)$, $\phi_2(p) \geq \phi_2(q)$. The result follows. **NW** case is symmetric. □

A.7. Extending value functions to extreme points

Value functions for the case when both **SE** and **NW** have a single essential coordinate were fully defined in Section A.5.2.3. Thus in what follows we will consider cases where **SE** or **NW** have two essential coordinates.

As indicated in (Wakker, 1991a), value functions might be driven to infinite values at the maximal/minimal points of rank-ordered subsets, nevertheless not implying existence of infinite standard sequences residing entirely within comonotonic cones. Put it another way, it might be not possible to “reach” a maximal/minimal point with a sequence lying entirely in **NW** or **SE**. Yet another way to say it is that for some maximal/minimal point z , the set $\mathbf{NW}_z(\mathbf{SE}_z)$ contains no standard sequences (see also (Wakker, 1991a) Remark 24).

The cornerstone of this section is Lemma 13. It plays the same role as proportionality of value functions plays in (Wakker, 1991a), effectively guaranteeing that both value functions ϕ_1 and ϕ_2 are limited if maximal/minimal elements exist.

Lemma 15. *Assume that **SE** has two essential coordinates. The following statements hold:*

- *If there exist a maximal $M_1 \in X_1$, ϕ_1 is bounded from above.*
- *If there exist a minimal $m_2 \in X_2$, ϕ_2 is bounded from below.*

*Assume that **NW** has two essential coordinates. The following statements hold:*

- *If there exist a minimal $M_1 \in X_1$, ϕ_1 is bounded from below.*
- *If there exist a maximal $m_2 \in X_2$, ϕ_2 is bounded from above.*

Proof. We shall only prove the first one. First, notice that there must exist $p \in X_2$ such that $M_1 p \in \mathbf{SE}$. Take $x_1 \in X_1$ and $v_2, w_2 \in X_2$ such that $v_2 \succ_2 w_2$, and $x_1 v_2 \in \mathbf{SE}$. If such points cannot be found, X has an additive representation (all $x \in \mathbf{NW}$), and the result follows. So we assume such points exist. By definition of $\mathbf{SE}_{x_1 v_2}$ it follows that $M_1 w_2, x_1 v_2, M_1 v_2 \in \mathbf{SE}$. Hence, we can evoke the argument from Wakker (1991a) Lemma 20.

If $M_1 w_2 \preceq x_1 v_2$ then we have an upper bound: $V_1(M_1) \leq V_1(x_1) + V_2(v_2) - V_2(w_2)$. If $M_1 w_2 \succ x_1 v_2$ then by monotonicity $M_1 v_2 \succ M_1 w_2 \succ x_1 v_2$ and hence exists $z_1 \in X_1$ such that $M_1 w_2 \sim z_1 v_2$, hence $z_1 v_2 \succ \beta w_2$ for all $\beta \in X_1$. \square

Lemma 16. *If $x_1 x_2 \in \Theta$ and $x_1 x_2$ is extreme, then*

$$\lim_{z \in \Theta, z_2 \rightarrow x_2} \phi_2(z_2) = \lim_{z \in \Theta, z_1 \rightarrow x_1} \phi_1(z_1).$$

Proof. For the case when **SE** or **NW** have two essential variables the result follows from Lemma 13, otherwise it is by definition of ϕ_i (see Section A.5.2.3). \square

A.7.1. Extending value functions to extreme elements of Θ

Extreme elements of Θ are the only representatives of maximal/minimal equivalence classes of $\mathbf{SE}(\mathbf{NW})$.

Lemma 17. *Let X_1 be essential on **SE**. If there exists $z \in \Theta$ such that z_2 is minimal, then $x \succ z$ for all $x \in \mathbf{SE}$. If X_2 is essential on **SE** and there exists $z \in \Theta$ such that z_1 is maximal, then $z \succ x$ for all $x \in \mathbf{SE}$. Similarly, if X_1 is essential on **NW** and there exists $z \in \Theta$ such that z_2 is maximal, then $z \succ x$ for all $x \in \mathbf{NW}$. If X_2 is essential on **NW** and there exists $z \in \Theta$ such that z_1 is minimal, then $x \succ z$ for all $x \in \mathbf{NW}$.*

Proof. We provide the proof just for one of the cases. Let **NW** have two essential variables. Assume z_2 is maximal. Since $z \in \Theta$, for all $x \in \mathbf{NW}$ holds $z_1 \succ_1 x_1$ and by maximality $z_2 \succ_2 x_2$. Hence, by Lemma 6, $z \succ x$ for all $x \in \mathbf{NW}$. The case with the minimal z_1 is symmetric. \square

Uniqueness of definition of ϕ_i at the extreme elements of Θ .

Lemma 18. *If both coordinates are essential on **SE** and **NW** the values of ϕ_i for extreme $x \in \Theta$ are uniquely defined. Moreover, $\phi_1(x_1) = \phi_2(x_2)$.*

Proof. Assume, for example $x_1x_2 \in \Theta$ and x_1 is minimal. Then any z_1x_2 such that $z_1 \succ_1 x_1$, belongs to **SE**, and any equivalence relation within **SE** involving z_1x_2 uniquely defines $\phi_2(x_2)$ (see (2)). Similarly, any x_1z_2 such that $z_2 \succ_2 x_2$, belongs to **NW**, and any equivalence relation within **NW** involving x_1z_2 uniquely defines $\phi_1(x_1)$. By Lemma 16 these values are equal. \square

Lemma 19. *If both coordinates are essential on **SE** but only one on **NW** (or vice versa) the values of ϕ_i for extreme $x \in \Theta$ can be set as follows:*

- *If $x_1x_2 \in \Theta$, **NW** has two essential coordinates and x_2 is maximal, then $\phi_2(x_2)$ is uniquely defined, $\phi_1(x_1)$ can be set to any value greater or equal to $\phi_2(x_2)$.*
- *If $x_1x_2 \in \Theta$, **NW** has two essential coordinates and x_1 is minimal, then $\phi_1(x_1)$ is uniquely defined, $\phi_2(x_2)$ can be set to any value less or equal to $\phi_1(x_1)$.*
- *If $x_1x_2 \in \Theta$, **SE** has two essential coordinates and x_2 is minimal, then $\phi_2(x_2)$ is uniquely defined, $\phi_1(x_1)$ can be set to any value less or equal to $\phi_2(x_2)$.*
- *If $x_1x_2 \in \Theta$, **SE** has two essential coordinates and x_1 is maximal, then $\phi_1(x_1)$ is uniquely defined, $\phi_2(x_2)$ can be set to any value greater or equal to $\phi_1(x_1)$.*

Proof. Consider the first case. $\phi_2(x_2)$ is defined uniquely as in the proof of Lemma 18. However, this is not possible for $\phi_1(x_1)$. This is because x is the only point in **NW** having x_1 as the first coordinate, and, by Lemma 17 there is no equivalence relation within **NW** which involves x . If X_1 is essential on **SE** then all points from the equivalence class which includes x also have x_1 as their first coordinate, which does not allow to elicit $\phi_1(x_1)$. If only X_2 is essential on **SE**, then the representations of equivalences involving x_1 do not include $\phi_1(x_1)$. \square

In the case where only one coordinate is essential on both **SE** and **NW** no special treatment is required for the extreme elements of Θ .

Lemma 20. *If $x \in \mathbf{SE}_{ext}$ then for any $y \in \mathbf{NW}$ such that $x \sim y$, we have:*

$$\phi^{NW}(x) = \phi^{NW}(y).$$

If further, x_1 is maximal, then $\phi^{SE}(x) > \phi^{SE}(y)$ for all $y \in \mathbf{SE}$. If x_2 is minimal, then $\phi^{SE}(x) < \phi^{SE}(y)$ for all $y \in \mathbf{SE}$.

If $x \in \mathbf{NW}_{ext}$ then for any $y \in \mathbf{SE}$ such that $x \sim y$, we have:

$$\phi^{SE}(x) = \phi^{SE}(y).$$

If further, x_1 is minimal, then $\phi^{NW}(x) < \phi^{NW}(y)$ for all $y \in \mathbf{NW}$. If x_2 is maximal, then $\phi^{NW}(x) > \phi^{NW}(y)$ for all $y \in \mathbf{NW}$.

Proof. In case when **SE** and **NW** have the same number of essential variables, value functions at the extreme points are defined uniquely (Lemma 18 for case when both variables are essential and by definition otherwise) and the second parts of each statement follow immediately by Lemma 17. If only **SE** or **NW** have two essential variables, the result follows by Lemma 19, Lemma 17, and definition of ϕ . \square

Lemma 21. *For any $x \in X$ we have:*

$$\phi_1(x_1) = \phi_2(x_2) \Rightarrow x \in \Theta.$$

Proof. Assume $\phi_1(x_1) = \phi_2(x_2)$ and $x \notin \mathbf{NW}$. By Lemma 7 exists $z \in \Theta$ such that $x \in \mathbf{SE}_z$. By structural assumption we have $\phi_2(z_2) \geq \phi_2(x_2) = \phi_1(x_1) \geq \phi_1(z_1)$, with at least one inequality being strict (otherwise $x = z$).

If z is non-extreme then by Lemma 13 we have $\phi_1(z_1) = \phi_2(z_2)$ - a contradiction. If z is extreme, the only cases when $\phi_2(z_2) > \phi_1(z_1)$ can hold is when either z_2 is minimal or z_1 is maximal. But it is easy to see that in this case the only points for which it is not possible to find a non-extreme z , are the extreme points themselves. \square

Lemma 22. *For all $x \in X$ such that $\phi_1(x_1) \geq \phi_2(x_2)$ we have $x \in \mathbf{SE}$. If $x \in X$ is such that $\phi_2(x_2) \geq \phi_1(x_1)$ then $x \in \mathbf{NW}$.*

Proof. For non-extreme points this follows from Lemma 14 and Lemma 21. Assume $\phi_1(x_1) \geq \phi_2(x_2)$. If $\phi_1(x_1) = \phi_2(x_2)$ then by Lemma 21 $x \in \Theta$, so we are done. Therefore, assume $\phi_1(x_1) > \phi_2(x_2)$. If $x \in \mathbf{NW}$, then by Lemma 14 it must be $\phi_2(x_2) \geq \phi_1(x_1)$, a contradiction. Therefore, $x \in \mathbf{SE}$. For extreme points the result follows from Lemma 19. \square

Finally, we can formulate:

Theorem 5. *The following statements hold:*

- If both **NW** and **SE** have two essential variables, then for all $x \in X$:

$$x \in \Theta \iff \phi_1(x_1) = \phi_2(x_2),$$

unless \succsim can be represented by an additive function (i.e $\lambda = 1$ in (2)).

- If only **NW** or only **SE** have two essential variables, then for all non-extreme $x \in X$:

$$x \in \Theta \Rightarrow \phi_1(x_1) = \phi_2(x_2),$$

while at extreme $x \in X$, $\phi_1(x_1)$ and $\phi_2(x_2)$ are related as in Lemma 19. Finally, for all $x \in X$:

$$\phi_1(x_1) = \phi_2(x_2) \Rightarrow x \in \Theta,$$

- If both **NW** and **SE** have only one essential variable, then for all $x \in X$:

$$x \in \Theta \iff \phi_1(x_1) = \phi_2(x_2).$$

Proof. Follows from Lemmas 13, 21, 18, 19. \square

A.8. Constructing a global representation on X

Lemma 23. Assume $z^1 \succ z^2$ for some $z^1, z^2 \in \Theta$. There exists z such that $z^1 \succ z \succ z^2$.

Proof. By order density and closedness assumption. \square

For all $x \in X$ let $\phi^x(x)$ be equal to $\phi^{SE}(x)$ if $x \in \mathbf{SE}$ or $\phi^{NW}(x)$ if $x \in \mathbf{NW}$. For points in Θ values of two latter functions coincide, so $\phi^x(x)$ is well-defined.

Lemma 24. Let $\phi^x(x) > \phi^y(y)$. Then, $x \succ y$.

Proof. If x and y belong to \mathbf{SE} or \mathbf{NW} the conclusion is immediate, so we only need to look at the remaining case. Assume $x \in \mathbf{SE}$, $y \in \mathbf{NW}$.

First we will show that it can't hold that $x \succ z, y \succ z$ or $z \succ x, z \succ y$ for all $z \in \Theta$. Assume $x \succ z, y \succ z$ for all $z \in \Theta$. Let $x_1 \succ_1 y_1$. If $x_1 y_2 \in \mathbf{SE}$, then exists $z_1 y_2 \in \Theta$ such that $x_1 y_2 \succ z_1 y_2 \succ y_1 y_2$, a contradiction. If $x_1 y_2 \in \mathbf{NW}$, then exists $x_1 z_2 \in \Theta$, such that $x_1 y_2 \succ x_1 z_2 \succ x_1 x_2$, again a contradiction. Other cases are symmetrical.

Hence, assume there exists $z^1 \in \Theta$, such that $x \succ z^1, y \succ z^1$ and $z^2 \in \Theta$ such that $z^2 \succ x$ or $z^2 \succ y$. The only non-trivial case is $z^2 \succ x \succ z^1, z^2 \succ y \succ z^1$ (in other cases one of the points z^1 or z^2 immediately leads to the conclusion). We have

$$\phi(z^2) > \phi(x) > \phi(y) > \phi(z^1),$$

hence also $\phi_1(z_1^2) = 0.5\phi(z^2) > 0.5\phi(x) > 0.5\phi(y) > 0.5\phi(z^1) = \phi_1(z_1^1)$. By denserangedness of ϕ_1 (see (Wakker, 1991a) equation (16)), we can find a point c_1 such that $0.5\phi(x) > \phi(c_1) > 0.5\phi(y)$. We have $c_1 z_2^1 \in \mathbf{SE}, c_1 z_2^2 \in \mathbf{NW}$, hence there exists c_2 such that $c_1 c_2 \in \Theta$. Since $c_1 c_2$ is not extreme, we have $\phi(c_1 c_2) = 2\phi_1(c_1)$, and hence $\phi(x) > \phi(c) > \phi(y)$. The first inequality is in \mathbf{SE} , while the second is in \mathbf{NW} , hence we conclude that $x \succ y$. \square

Lemma 25. Let $x \succ y$. Then, $\phi(x) > \phi(y)$.

Proof. By Lemma 24 we have $x \succ y \Rightarrow \phi(x) \geq \phi(y)$. Hence, we need to show that $\phi(x) \neq \phi(y)$. Assume, $x \in \mathbf{SE}, y \in \mathbf{NW}$. If \mathbf{SE} or \mathbf{NW} have only one essential coordinate, then by Lemma 4 exists $z \in \Theta$, equivalent either to x or y , from which the conclusion is immediate. Hence, assume both areas have two essential coordinates.

x_1 and x_2 can't be both minimal, because otherwise $x \succ y$ cannot hold by pointwise monotonicity, so assume x_1 is not minimal. We will find a point z in \mathbf{SE} such that $x \succ z \succ y$. Take some z_1 such that $x_1 \succ_1 z_1$ and $z_1 x_2 \in \mathbf{SE}$ (it can be found by closedness and order density). If $z_1 x_2 \succ y$, we have $\phi(x) > \phi(z_1 x_2) \geq \phi(y)$, otherwise by restricted solvability we can find w_1 such that $w_1 x_2 \sim y$, $w_1 \in \mathbf{SE}$, and hence $\phi(x) > \phi(w_1 x_2) = \phi(y)$ (equality follows from Lemma 24). The case when x_2 is not maximal is identical. \square

Theorem 6. For any $x, y \in X$ we have

$$x \succ y \iff \phi(x) \geq \phi(y).$$

Proof. Immediate by Lemmas 24 and 25. \square

A.9. The representation is a Choquet integral

Representations ϕ^{SE} and ϕ^{NW} uniquely define a capacity ν . For the case when **SE** or **NW** has two essential coordinates, set (using (2)):

$$\begin{aligned}\nu(\{1\}) &= \frac{1}{1+k} && (\text{from } \phi^{SE}) \\ \nu(\{2\}) &= \frac{\lambda}{1+\lambda k} && (\text{from } \phi^{NW}) \\ \nu(\{1, 2\}) &= 1.\end{aligned}$$

Thus, we obtain

$$\begin{aligned}C(\nu, \phi(x)) &= \phi^{SE}(x) = \frac{1}{1+k}\phi_1(x_1) + \frac{k}{1+k}\phi_2(x_2), && \text{for all } x \in \mathbf{SE}, \\ C(\nu, \phi(x)) &= \phi^{NW}(x) = \frac{1}{1+\lambda k}\phi_1(x_1) + \frac{\lambda k}{1+\lambda k}\phi_2(x_2), && \text{for all } x \in \mathbf{NW}.\end{aligned}$$

Assume now, that **SE** and **NW** has only one essential coordinate. If X_1 is essential on **SE** set $\nu(1) = 1$, otherwise zero. If X_2 is essential on **NW** set $\nu(2) = 1$, otherwise zero. As above, set $\nu(\{1, 2\}) = 1$ We obtain:

$$\begin{aligned}C(\nu, \phi(x)) &= \phi_1(x_1), && \text{if } X_1 \text{ is essential on the area containing } x, \\ C(\nu, \phi(x)) &= \phi_2(x_2), && \text{if } X_2 \text{ is essential on the area containing } x,\end{aligned}$$

in particular $C(\nu, \phi(x)) = \max(\phi_1(x_1), \phi_2(x_2))$ if X_1 is essential on **SE**, X_2 is essential on **NW**, $C(\nu, \phi(x)) = \min(\phi_1(x_1), \phi_2(x_2))$ if X_2 is essential on **SE**, X_1 is essential on **NW**.

A.10. Uniqueness

Uniqueness properties are similar to those obtained in the homogeneous case $X = Y^n$, but are modified to accommodate for the heterogeneous structure of the set X in this paper.

Lemma 26. *Representation (1) has the following uniqueness properties:*

1. *If $\mathbf{NW} = \mathbf{SE} = X$ then for any functions $g_1 : X_1 \rightarrow \mathbb{R}, g_2 : X_2 \rightarrow \mathbb{R}$ such that (1) holds with f_i substituted by g_i , we have $f_i(x_i) = \alpha g_i(x_i) + \beta$.*
2. *If both coordinates are essential on **NW** and **SE**, then for any functions $g_1 : X_1 \rightarrow \mathbb{R}, g_2 : X_2 \rightarrow \mathbb{R}$ such that (1) holds with f_i substituted by g_i , we have $f_i(x_i) = \alpha g_i(x_i) + \beta$.*
3. *If both coordinates are essential on **NW**, but only one coordinate is essential on **SE**, then for any functions $g_1 : X_1 \rightarrow \mathbb{R}, g_2 : X_2 \rightarrow \mathbb{R}$ such that (1) holds with f_i substituted by g_i , we have :*
 - $f_i(x_i) = \alpha g_i(x_i) + \beta$, for all x such that $f_1(x_1) < \max f_2(x_2)$ and $f_2(x_2) > \min f_1(x_1)$;
 - $f_i(x_i) = \phi_i(g_i(x_i))$ where ϕ_i is an increasing function, otherwise.

4. If both coordinates are essential on **SE**, but only one coordinate is essential on **NW**, then for any functions $g_1 : X_1 \rightarrow \mathbb{R}, g_2 : X_2 \rightarrow \mathbb{R}$ such that (1) holds with f_i substituted by g_i , we have :

- $f_i(x_i) = \alpha g_i(x_i) + \beta$, for all x such that $f_2(x_2) < \max f_1(x_1)$ and $f_1(x_1) > \min f_2(x_2)$;
- $f_i(x_i) = \psi_i(g_i(x_i))$ where ψ_i is an increasing function, otherwise.

5. If one coordinate is essential on **NW** and another one on **SE**, then for any functions $g_1 : X_1 \rightarrow \mathbb{R}, g_2 : X_2 \rightarrow \mathbb{R}$ such that (1) holds with f_i substituted by g_i , we have : $f_i(x_i) = \psi_i(g_i(x_i))$ where ψ_i are increasing functions such that $f_1(x_1) = f_2(x_2) \iff g_1(x_1) = g_2(x_2)$.

Proof. 1. Direct by uniqueness properties of additive representations.

2. Direct by uniqueness properties of additive representations, Lemma 18, Theorem 6.

3. Assume **NW** has two essential coordinates. If there exists an element in Θ such that x_1 is minimal or x_2 is maximal, then, by Lemma 15, there exist respectively a minimal $\phi_1(x_1)$ and maximal $\phi_2(x_2)$. Points x such that $f_1(x_1) < \max f_2(x_2)$ and $f_2(x_2) > \min f_1(x_1)$ are precisely the elements for which there exist $a \in X_1$ or $p \in X_2$ such that either $ax_2 \in \mathbf{NW}$ or $x_1p \in \mathbf{NW}$. From this follows that uniqueness of ϕ_i for these points is defined by the uniqueness properties of **NW** and definition of ϕ_i on **SE**, i.e. $f_i(x_i) = \alpha g_i(x_i) + \beta$. For the remaining points (including extreme elements of Θ), uniqueness is derived from the uniqueness of ordinal representations and Lemma 4.

4. The proof is identical to the one in the previous point.

5. Uniqueness properties are derived from the uniqueness of ordinal representations, Lemma 4 and definition of ϕ_i (Section A.5.2.3).

□

Uniqueness part of Theorem 1 directly follows from Lemma 26.

A.11. Necessity of axioms

A4.. Let $ap \preccurlyeq bq, ar \succcurlyeq bs, cp \succcurlyeq dq$ and assume $cr \prec ds$. Let also $ap, bq, cp, dq \in \mathbf{NW}$, $ar, bs, cr, ds \in \mathbf{SE}$ and X_1 to be symmetric in **NW** (the other cases are symmetric). We obtain:

$$\begin{aligned} \alpha_1 f_1(a) + \alpha_2 f_2(p) &\leq \alpha_1 f_1(b) + \alpha_2 f_2(q) \\ \alpha_1 f_1(c) + \alpha_2 f_2(p) &\geq \alpha_1 f_1(d) + \alpha_2 f_2(q) \\ \beta_1 f_1(a) + \beta_2 f_2(r) &\geq \beta_1 f_1(b) + \beta_2 f_2(s) \\ \beta_1 f_1(c) + \beta_2 f_2(r) &< \beta_1 f_1(d) + \beta_2 f_2(s) \end{aligned}$$

From the first two inequalities and essentiality of X_1 ($\alpha_1 \neq 0$) follows $f_1(a) + f_1(d) \leq f_1(b) + f_1(c)$. Last two inequalities imply $f_1(a) + f_1(d) > f_1(b) + f_1(c)$, a contradiction.

We also give the “necessity” proof of the condition in Lemma 3, since comparing it with the necessity proof of **A5** allows to elicit some interesting implications of essentiality.

Lemma 3. Let $\{g_1^{(i)} : g_1^{(i)}y_0 \sim g_1^{(i+1)}y_1, g_1^{(i)} \in X_1, i \in N\}$ and $\{h_2^{(i)} : x_0h_2^{(i)} \sim x_1h_2^{(i+1)}, h_2^{(i)} \in X_x, i \in N\}$ be two standard sequences, the first entirely contained in **NW** and the second in **SE**. Assume also, that there exist $z_1, z_2 \in X$, $p, q \in X_2$, $a, b \in X_1$ such that $g_1^{(i)}p, g_1^{(i)}q \in \mathbf{NW}$, and $ah_2^{(i)}, bh_2^{(i)} \in \mathbf{SE}$ for all i , and $g_1^{(i)}p \sim bh_2^{(i)}$ and $g_1^{(i+1)}p \sim bh_2^{(i+1)}$. Finally, assume, $g_1^{(i+2)}p \succ bh_2^{(i+2)}$. Other cases are symmetric.

$$\begin{aligned}\alpha_1 f_1(g_1^{(i)}) + \alpha_2 f_2(y_0) &= \alpha_1 f_1(g_1^{(i+1)}) + \alpha_2 f_2(y_1) \\ \alpha_1 f_1(g_1^{(i)}) + \alpha_2 f_2(y_0) &= \alpha_1 f_1(g_1^{(i+1)}) + \alpha_2 f_2(y_1)\end{aligned}$$

$$\begin{aligned}\beta_1 f_1(x_0) + \beta_2 f_2(h_1^{(i)}) &= \beta_1 f_1(x_1) + \beta_2 f_2(h_1^{(i+1)}) \\ \beta_1 f_1(x_0) + \beta_2 f_2(h_1^{(i+1)}) &= \beta_1 f_1(x_1) + \beta_2 f_2(h_1^{(i+2)})\end{aligned}$$

$$\begin{aligned}\alpha_1 f_1(g_1^{(i)}) + \alpha_2 f_2(p) &= \beta_1 f_1(b) + \beta_2 f_2(h_1^{(i)}) \\ \alpha_1 f_1(g_1^{(i+1)}) + \alpha_2 f_2(p) &= \beta_1 f_1(b) + \beta_2 f_2(h_1^{(i+1)})\end{aligned}$$

First two equations imply $\alpha_1(f_1(g_1^{(i)}) - f_1(g_1^{(i+1)})) = \alpha_1(f_1(g_1^{(i+1)}) - f_1(g_1^{(i+2)}))$. The following two imply $\beta_1(f_2(h_1^{(i)}) - f_2(h_1^{(i+1)})) = \beta_1(f_2(h_1^{(i+1)}) - f_2(h_1^{(i+2)}))$. Finally, the last two equations imply $\alpha_1(f_1(g_1^{(i)}) - f_1(g_1^{(i+1)})) = \beta_1(f_2(h_1^{(i)}) - f_2(h_1^{(i+1)}))$. Apparently $\alpha_1(f_1(g_1^{(i+1)}) - f_1(g_1^{(i+2)})) < \beta_1(f_2(h_1^{(i+1)}) - f_2(h_1^{(i+2)}))$ is then a contradiction.

If we were to add an essentiality condition to Lemma 3, the statement can be made stronger as shown below.

A5.. Assume $ap \preccurlyeq bq$, $cp \succcurlyeq dq$ and $ay_0 \sim x_0\pi(a)$, $by_0 \sim x_0\pi(b)$, $cy_1 \sim x_1\pi(c)$, $dy_1 \sim x_1\pi(d)$, and also $e\pi(a) \succcurlyeq g\pi(b)$. Also, X_1 is essential on the set (**NW** or **SE**) which includes ap, bq, cp, dq , and X_2 is essential on the set (**NW** or **SE**), which includes $x_0\pi(a)$ and $x_0\pi(b)$. Finally, assume $e\pi(c) \prec g\pi(d)$.

We get

$$\begin{aligned}\alpha_1 f_1(a) + \alpha_2 f_2(p) &\leq \alpha_1 f_1(b) + \alpha_2 f_2(q) \\ \alpha_1 f_1(c) + \alpha_2 f_2(p) &\geq \alpha_1 f_1(d) + \alpha_2 f_2(q)\end{aligned}$$

$$\begin{aligned}\beta_1 f_1(e) + \beta_2 f_2(\pi(a)) &\geq \beta_1 f_1(g) + \beta_2 f_2(\pi(b)) \\ \beta_1 f_1(e) + \beta_2 f_2(\pi(b)) &< \beta_1 f_1(g) + \beta_2 f_2(\pi(d))\end{aligned}$$

$$\begin{aligned}\gamma_1 f_1(a) + \gamma_2 f_2(y_0) &= \delta_1 f_1(x_0) + \delta_2 f_2(\pi(a)) \\ \gamma_1 f_1(b) + \gamma_2 f_2(y_0) &= \delta_1 f_1(x_0) + \delta_2 f_2(\pi(b))\end{aligned}$$

$$\begin{aligned}\gamma_1 f_1(c) + \gamma_2 f_2(y_1) &= \delta_1 f_1(x_1) + \delta_2 f_2(\pi(c)) \\ \gamma_1 f_1(d) + \gamma_2 f_2(y_1) &= \delta_1 f_1(x_1) + \delta_2 f_2(\pi(d))\end{aligned}$$

First two inequalities and the essentiality of X_1 ($\alpha_1 \neq 0$) imply $f_1(a) - f_1(b) \leq f_1(c) - f_1(d)$. Second pair of inequalities yields $f_2(\pi(c)) - f_2(\pi(d)) < f_2(\pi(a)) - f_2(\pi(b))$,

while the final pair of equations leads to $\gamma_1(f_1(c) - f_1(d)) = \delta_2(f_2(\pi(c)) - f_2(\pi(d)))$. Combining these results and due to essentiality of X_2 (hence $\delta_2 \neq 0$) we get:

$$\gamma_1(f_1(a) - f_1(b)) \leq \gamma_1(f_1(c) - f_1(d)) = \delta_2(f_2(\pi(c)) - f_2(\pi(d))) < \delta_2(f_2(\pi(a)) - f_2(\pi(b))),$$

which contradicts the third pair of inequalities above, which yield $\gamma_1(f_1(a) - f_1(b)) = \delta_2(f_2(\pi(a)) - f_2(\pi(b)))$.

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